

Ratio of change and Derivative

$$y = f(x)$$

$$\downarrow \qquad \downarrow$$

$$\Delta y \qquad \Delta x$$

$\frac{\Delta y}{\Delta x}$ = average rate of change

$$\Delta x \rightarrow 0$$

$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$ = instantaneous rate of change

$\frac{dy}{dx} > 0$ ↑ antities increasing

$\frac{dy}{dx} < 0$ ↓ antities decreasing

Q. A balloon which always remains spherical, has a variable radius. Find the rate at which its volume is increase wrt its radius when the radius is 7 cm.

Ans:- Let r = radius of the spherical balloon

V = Volume of the spherical balloon

$$V = \frac{4}{3}\pi r^3$$

Diffr. wrt r

$$\frac{dv}{dr} = \frac{4}{3}\pi \times 3r^2$$

$$\frac{dv}{dr} \Big|_{r=7} = 4\pi(7)^2 = 196\pi \text{ cm}^3/\text{cm}$$

- Q. The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue when $x = 5$.

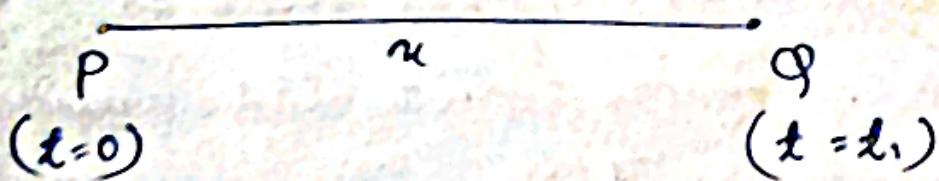
$$\frac{dR}{dx} \Big|_{x=5} = 6x + 36$$

$$= 6(5) + 36$$

$$= 6(5+6)$$

$$= 6 \times 11 = 66$$

- Q. A car starts from a point P at time $t=0$ & stops at point Q . The distance x , in m, covered by it in t sec, is given by $x = t^2(2 - t/3)$. Find the time taken by it to reach Q . Also find distance PQ .



$$\frac{dx}{dt} = 4t - \frac{3t^2}{3}$$

At Q , $\frac{dx}{dt} = 0 \quad . \quad t = t_1$,

$$\Rightarrow 4t_1 - t_1^2 = 0$$

$$\Rightarrow t_1(4 - t_1) = 0$$

$$\Rightarrow t_1 = 0 \quad \text{or} \quad t_1 = 4$$

$t_1 = 4 \text{ sec}$

car stops

at $x=4$

$$a = 4^2 (2 - \frac{4}{3})$$

$$= 16 (\frac{2}{3})$$

$$= \frac{32}{3} \text{ m} \quad (\text{Ans})$$

Q. If x & y are the sides of two squares

$$\text{S.t } y = x - x^2$$

Find the change of the area of second square w.r.t the area of the first square.

$$A_1 = \text{area of the 1st square} = x^2$$

$$A_2 = \text{area of the 2nd square} = y^2$$

$$y^2 = (x - x^2)^2$$

$$\frac{dA_2}{dA_1} = \frac{dA_2/dx}{dA_1/dx}$$

$$= \frac{x(x-x^2)(1-2x)}{2x}$$

$$= \frac{x(1-x)(1-2x)}{x}$$

$$= 1 - 3x + 2x^2 \quad (\text{Ans})$$

Q. If the area of circle is increasing at a uniform rate, then prove that the perimeter varies inversely as the radius.

Let,

r = radius of the circle

A = area of the circle

P = Perimeter of the circle

$$\frac{dA}{dt} = k \quad (k > 0)$$

We have to show

$$\frac{dP}{dt} \propto \frac{1}{r}$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{k}{2\pi r}$$

$$P = 2\pi r$$

$$\begin{aligned}\frac{dP}{dt} &= 2\pi \frac{dr}{dt} \\ &= 2\pi \times \frac{k}{2\pi r} = \frac{k}{r}\end{aligned}$$

$$\Rightarrow \frac{dP}{dt} \propto \frac{1}{r}$$

Q. A spherical ball of salt is dissolving in water at the rate of decreasing in volume at any instant is proportional to the surface prove that radius is decreasing at a constant rate.

Let r = radius of the ball

V = volume of the ball

A = surface " " "

$$\frac{dV}{dt} \propto -A$$

$$\Rightarrow \frac{dV}{dt} = -kA$$

$$\frac{d}{dt} \left(\frac{4}{3}\pi r^3 \right) = -k(4\pi r^2)$$

$$\frac{4}{3}\pi \times 3r^2 \frac{dr}{dt} = -4k\pi r^2$$

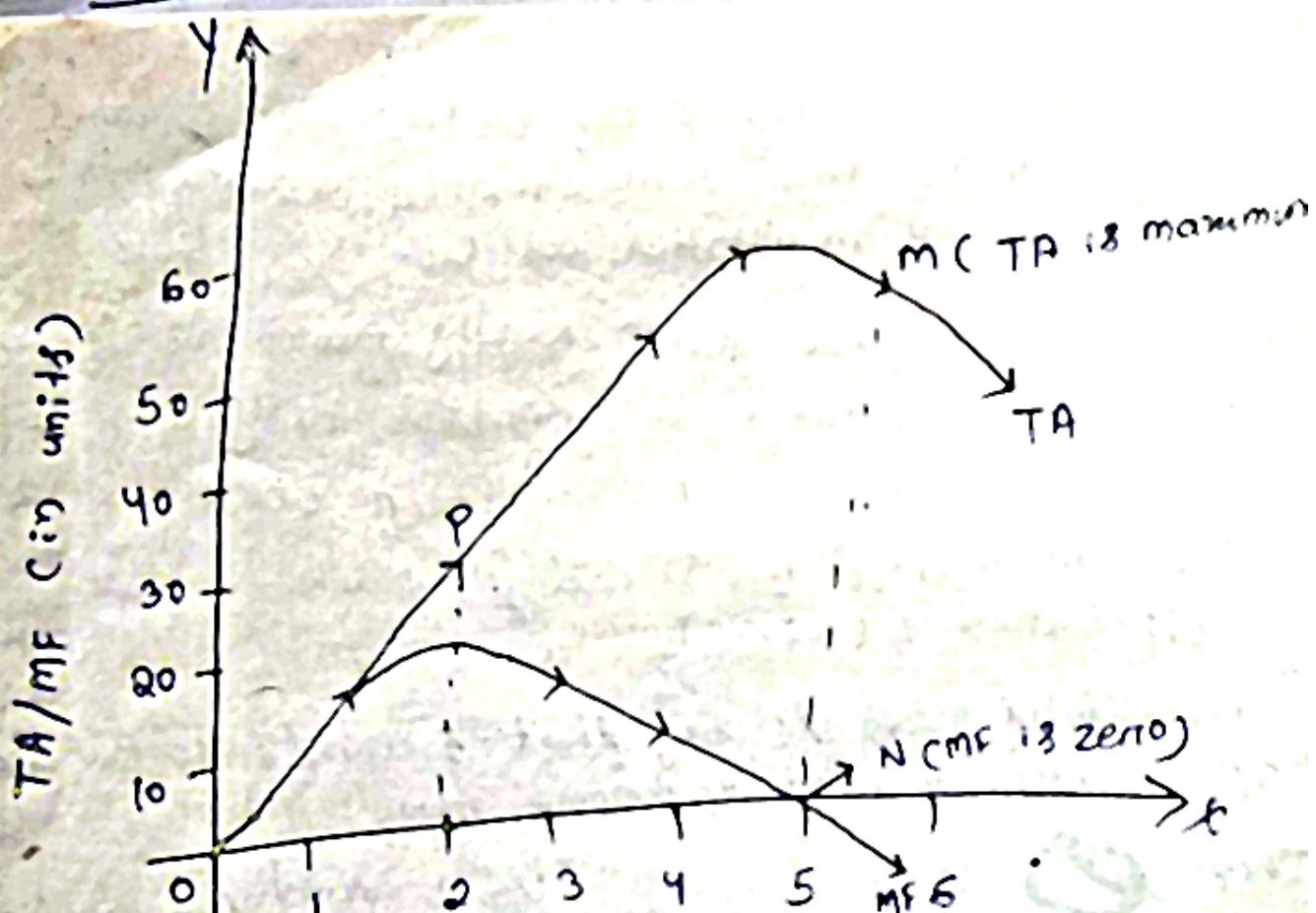
$$\frac{dr}{dt} = -k$$

r is decreasing

Application Relationship betⁿ total average & marginal function :-

- As long a total average increases at increasing rate marginal function also increases.
- When total average increase at diminishing rate, marginal function decreased.
- When total average reaches its maximum point marginal function become zero.
- When total average starts decreasing, marginal function becomes negative.

fixed factor (in acres)	units of variable factor (labour)	TA (units)	MF (units)
1	0	0	-
1	1	10	10
1	2	30	20
1	3	45	15
1	4	52	7
1	5	52	0
1	6	48	4



units of variable factor

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Continuity & Differentiability of a function

Let $f(x)$ be a real valued function defined on (a, b) .

Let $c \in (a, b)$

Then $f(x)$ is differentiable at $x=c$

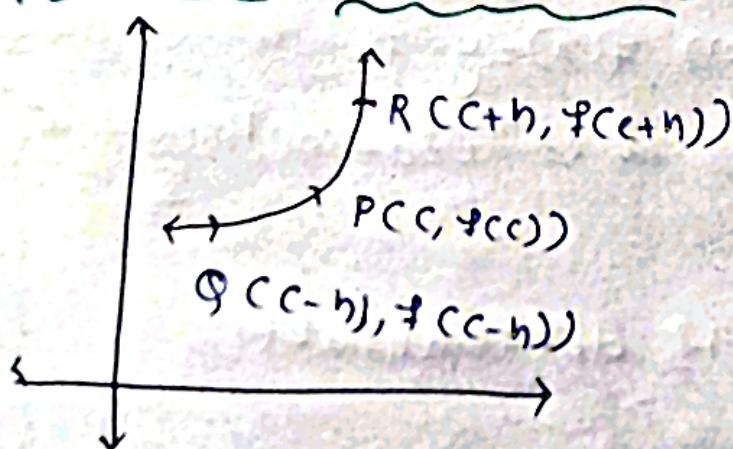
$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exist finitely

or $\lim_{c-h \rightarrow 0} \frac{f(c-h) - f(c)}{c-h - c} = \lim_{c+h \rightarrow 0} \frac{f(c+h) - f(c)}{c+h - c}$

Put $x = c-h$ L.H.S

$x = c+h$ R.H.S

Geometrical Interpretation



$$\begin{aligned}\text{Slope of } QR &= \frac{f(c+h) - f(c-h)}{c+h - c} \\ &= \frac{f(c+h) - f(c)}{h}\end{aligned}$$

as $h \rightarrow 0$ $Q \rightarrow P$

$$\lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h} \quad \text{tangent at } P$$

$$\text{Slope of PR} = \frac{f(c+h) - f(c)}{c+h - c}$$

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

Differentiability at any interval

A function $f(x)$ is differentiable/dervable if

$f(x)$ is differentiable $\forall x \in (a, b)$.

- (1) $f(x)$ is differentiable in (a, b)
- (2) $f(x)$ is differentiable at the end point a & b .

Differentiability continuity

if a function $f(x) = y$ is differentiable at $x=a$, then it is also continuous at $x=a$.

$\because y = f(x)$ is differentiable

at $x=a$

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a) \quad (1)$$

$$\text{Now, } f(x) = \{ f(x) - f(a) \} + f(a)$$

$$= \left(\frac{f(x) - f(a)}{x - a} \right) \cdot (x-a) + f(a)$$

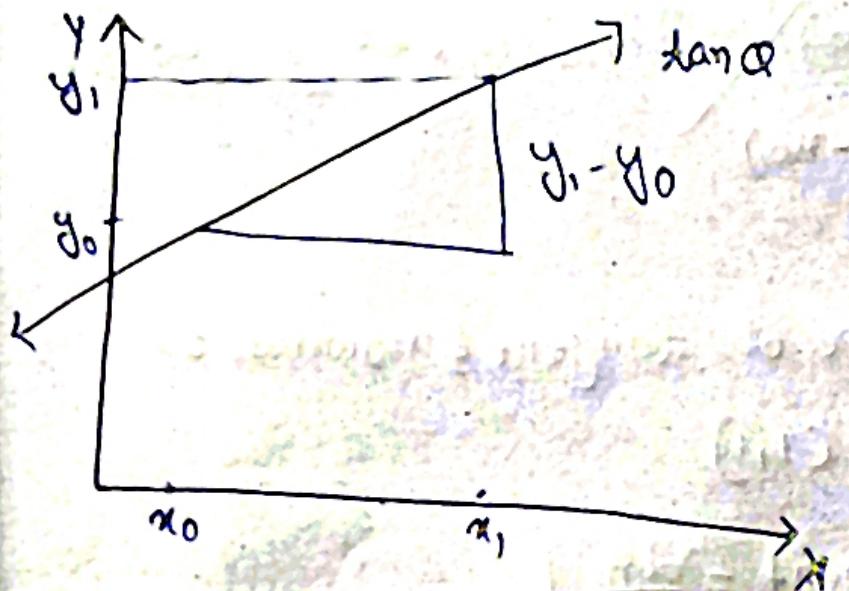
$$\lim_{x \rightarrow a} f(x)$$

$$= \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x-a) + f(a) \right)$$

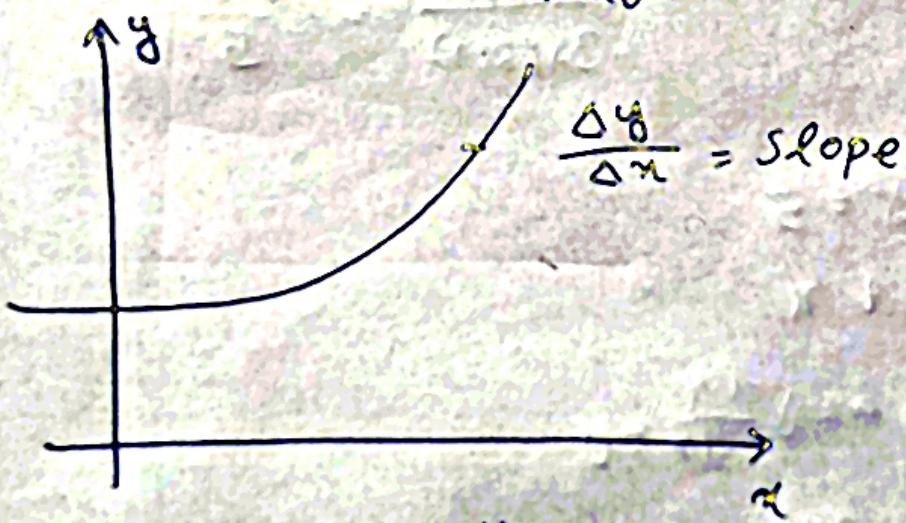
$$= f'(a) \cdot 0 + f(a)$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = f(a)$$

Derivative as slope of a curve



$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$



$$\text{Average} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of tangent

$$\frac{dy}{dx} \Big|_{(x_0, y_0)}$$

Find the slope of tangent & normal to

$$x^2 + 3y + y^2 = 5 \text{ at } (1, 1)$$

$$\text{Slope of tangent} = \frac{dy}{dx} \Big|_{(1, 1)}$$

$$= \frac{-2(1)}{3+2(1)} = -\frac{2}{5}$$

$$x^2 + 3y + y^2 = 5$$

Dif^f. w.r.t x

$$2x + 3\frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{3+2y}$$

$$\text{Slope of normal} = 5/2$$

Prove that tangents to $y = x^2 - 5x + 6$ at $(2,0)$ & $(3,0)$ are at right angle.

$$y = x^2 - 5x + 6$$

$$m_1, m_2 = -1$$

$$\frac{dy}{dx} = 2x - 5$$

$$m_1 = \left. \frac{dy}{dx} \right|_{(2,0)} = 4 - 5 = -1$$

$$m_2 = \left. \frac{dy}{dx} \right|_{(3,0)} = 6 - 5 = 1$$

Q Find the points on the curve $y = x^3 - 2x^2 - x$ at which the tangents are parallel to the line $y = 3x - 2$

$$y = x^3 - 2x^2 - x$$

Diff. w.r.t x

$$\frac{dy}{dx} = 3x^2 - 4x - 1$$

Let (x_0, y_0) be the pt of contact

$$\text{Slop of tangent} = 3x_0^2 - 4x_0 - 1$$

$$\text{Slop of line} = 3$$

$$\text{ATQ}, 3x_0^2 - 4x_0 - 1 = 3$$

$$\Rightarrow 3x_0^2 - 4x_0 - 4 = 0$$

$$\Rightarrow 3x_0^2 - 6x_0 + 2x_0 - 4 = 0$$

$$\Rightarrow 3x_0(x_0 - 2) + 2(x_0 - 2) = 0$$

$$x_0 = 2 \text{ or } x_0 = -2/3$$

$$x_0 = 2, y_0 = 3 - 3 - 2 = -2$$

$$(2, -2)$$

$$x_0 = -2/3, y_0 = \left(-\frac{2}{3}\right)^3 - 2(-2/3)^2 + 2/3$$

$$= -\frac{8}{27} + \frac{8}{9} + \frac{2}{3} = -\frac{14}{27}$$

$$\left(-\frac{2}{3}, -\frac{14}{27}\right)$$