BGD COLLEGE ,KESAIBAHAL

Blended learning modules

1st Year 1st SEM

Subject and paper :- **MECHANICS (PAPER-II)**

Contents

Elastic

Elastic Limit:

When an external force acts on a body, the body tends to undergo some deformation. If the external force is removed and the body comes back to its origin shape and size, the body is known as elastic body. This property, by virtue of which certain materials return back to their original position after the removal of the external force, is called elasticity.

The body will regain its previous shape and size only when the deformation caused by the external force, is within a certain limit. Thus there is a limiting value of force up to and within which, the deformation completely disappears on the removal of the force. The value of stress corresponding to this limiting force is known as the elastic limit of the material.

Hooke's law:

It states that when a material is loaded within elastic limit, the stress is proportional to the strain. This means the ratio of the stress to the corresponding strain is a constant within the elastic limit.

Stress /Strain = Constant

This constant is known as elastic constant.

Types of Elastic Constants:

There are three elastic constants;

Normal stress/ Normal strain = **Young'smodulus** or Modulus of elasticity (E)

Shear stress/ Shear strain = **Shear modulus** or Modulus of Rigidity (G)

Direct stress/ Volumetric strain = **Bulk modulus** (B)

Young's Modulus or Modulus of elasticity (E):

It is defined as the ratio of normal stress (σ) to the longitudinal strain (e).

$$E = \text{Stress/Strain} = (\sigma_n) / (e)$$

Modulus of Rigidity or Shear Modulus (η):

It is the ratio between shear stress (**τ**) and shear strain (**e_s**). It is denoted by G or C.η

$$\eta = \tau/e_s$$

Bulk Modulus or Volume Modulus of Elasticity (K):

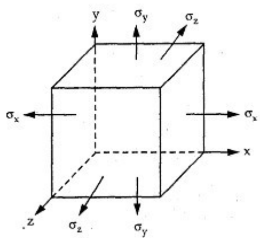
It may be defined as the ratio of normal stress (on each face of a solid cube) to volumetric strain. It is denoted by K. Bulk modulus is a measure of the resistance of a material to change of volume without change of shape or form.

$$K = \text{Direct Stress / Volumetric strain}$$

$$= \sigma/e_v$$

Relation between E, K and Poisson's Ratio (μ or 1/m):

Consider a cubical element subjected to volumetric stress σ which acts simultaneously along the mutually perpendicular x, y and z-direction.



The resultant strains along the three directions can be worked out by taking the effect of individual stresses.

Strain in the x-direction,

e_x = strain in x-direction due to σ_x - strain in x-direction due to σ_Y - strain in x-direction due to σ_Z

$$= \sigma_x/E - \sigma_y/mE - \sigma_z/mE \text{ -------------------(1)}$$

$$\text{But } \sigma_X = \sigma_Y = \sigma_Z = \sigma$$

$$\text{So } \sigma_X = \sigma/E - \sigma/mE - \sigma/mE$$

$$\sigma_X = \sigma/E(1-2/m)$$

$$\text{Similarly } \sigma_Y = \sigma/E\,(1-2/m) \text{ and } \sigma_Z = \sigma/E\,(1-2/m)$$

Now Volumetric strain

$$e_V = e_X + e_Y + e_Z = 3\sigma/E(1–2/m) \text{ ----------(2)}$$

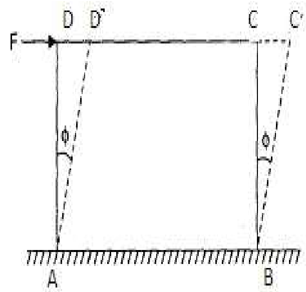
Since bulk modulus $K = \sigma/ e_V$

$$K = \sigma/ \{3\sigma/E\ (1–2/m)\ \}$$

On simplification **E = 3K (1-2/m) or E = 3K (1-2μ)**

Relation between E, η and Poisson's Ratio (1/m or μ):

Consider a cubic element *ABCD* fixed at the bottom face and subjected to shearing force at the top face. The block experiences the following effects due to this shearing load:



1. Shearing stress τ is induced at the faces DC and *AB*

2. Complimentary shearing stress of the same magnitude is set up on the faces *AD* and BC.

3. The block distorts to a new configuration ABC'D'.

4. The diagonal AC elongates (tension) and diagonal *BD* shortens (compression).

Longitudinal strain 'e' in diagonal AC = (AC' – AC)/AC

= (AC' – AE)/AC

= EC'/AC ----------- (1)

Where CE is perpendicular from C onto AC'.

Since CC' is too small, assume

Angle ACB = Angle ACB = 45^0

Therefore EC' = CC'cos45^0 = CC'/√2

Longitudinal strain 'e' = CC'/AC√2

= CC'/√2.BC.√2

= tanΦ/2 = Φ/2 = e_S/2 ---------------- (2)

Where, Φ = CC'/BC represents the shear strain (e_S)

In terms of shear stress τ and modulus of rigidity η, shear strain (e_S) = τ/η-------------(3)

Putting shear strain (e_S) = 2. Longitudinal strain Longitudinal strain of diagonal AC = $\tau/2$ -------------------------------------- (4)

The strain in diagonal AC is also given by

= strain due to tensile stress in AC - strain due to compressive stress in *BD*

$= \tau/E - (-\tau/mE) = \tau/E\ (1 + 1/m)$ ------------- (5)

From equation (4) and (5), we get

$$\tau/2\eta = \tau/E(1 + 1/m)$$

or $\mathbf{E = 2\eta\ (1 + 1/m)}$ or $\mathbf{E = 2\eta\ (1+\mu)}$------------ (6)

Relation between E, η and K:

With reference to the relations (1) and (6) derived above,

$$E = 2\eta\ (1 + \mu) = 3K\ (1- 2\mu)$$

To eliminate 1/m from these two expressions for E, we have

$$\mathbf{E = 9K\eta / (G + 3K)}$$

Finally; $\mathbf{E = 2\eta\ (1 + \mu) = 3K\ (1 - 2\mu)}$ **or** $\mathbf{E = 9K\eta / (\eta + 3K)}$

Or $\mathbf{1/E = 1/9K + 1/3\eta}$

❖ Twisting couple on a cylinder or wire:

Consider a cylindrical rod of length l radius r and coefficient of rigidity η. Its upper end is fixed and a couple is applied in a plane perpendicular to its length at lower end as shown in fig.(a)

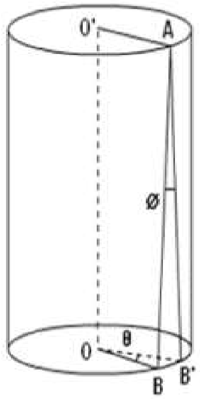


Figure : (a)

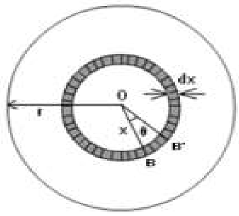


Figure : (b)

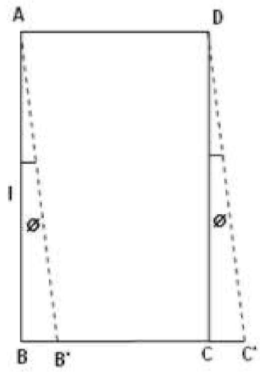


Figure : (c)

Consider a cylinder is consisting a large number of co-axial hollow cylinder. Now, consider a one hollow cylinder of radius x and radial thickness dx as shown in fig.(b). Let θ is the twisting angle. The displacement is greatest at the rim and decreases as the center is approached where it becomes zero.

As shown in fig.(a), Let AB be the line parallel to the axis OO′ before twist produced and on twisted B shifts to B′, then line AB become AB′.

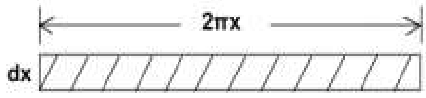
Before twisting if hollow cylinder cut along AB and flatted out, it will form the rectangular ABCD as shown in fig.(c). But if it will be cut after twisting it takes the shape of a parallelogram AB′C′D.

The angle of shear $\angle BAB' = \emptyset$

From fig.(c) $BB' = l\emptyset$

From fig.(b) $BB' = x\theta$

$\therefore l\emptyset = x\theta$



$$\therefore \ \emptyset = \frac{x\theta}{l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The modules of rigidity is

$$\eta = \frac{Shearing\ spress}{angle\ of\ shear} = \frac{F}{\phi}$$

$$\therefore \ F = \eta \cdot \emptyset = \frac{\eta x\theta}{l}$$

The surface area of this hollow cylinder $= 2\pi x dx$

∴ Total shearing force on this area

$$= 2\pi x dx \cdot \frac{\eta x\theta}{l}$$

$$= 2\pi\eta \frac{\theta}{l} x^2 dx$$

The moment of this force

$$= 2\pi\eta \frac{\theta}{l} x^2 dx \cdot x$$

$$= \frac{2\pi\eta\theta}{l} x^3 \cdot dx$$

Now, integrating between the limits x = 0 and x = r,
We have, total twisting couple on the cylinder

$$= \int_o^r \frac{2\pi\eta\theta}{l} x^3 dx$$

$$= \frac{2\pi\eta\theta}{l} \int_o^r x^3 dx$$

$$= \frac{2\pi\eta\theta}{l} \left[\frac{x^4}{4}\right]_o^r$$

∴ Total twisting couple

$$= \frac{\pi\eta\theta r^4}{2l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Then, the twisting couple per unit twist ($\theta = 1$) is

$$C = \frac{\pi\eta r^4}{2l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

This twisting couple per unit twist is also called the torsional rigidity of the cylinder or wire.

BENDING OF BEAMS

A beam is a rod or a bar of uniform cross section of homogeneous and isotropic elastic material whose length is large compared to its thickness.

2.1 Basic Assumptions involved in the simple theory of bending:

The cross section of the beam remains unaltered during best so that that shearing stresses over any section are negligibly small.

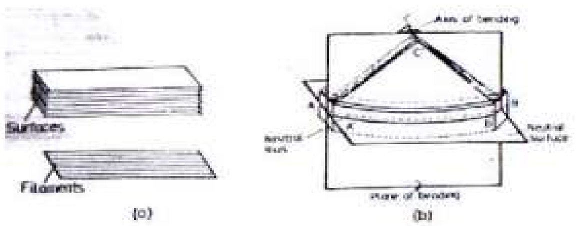
The radius of curvature of bent beam is large compared with its thickness.

The minimum deflection of the beam is small compared with its length and

The Young's modulus of the beam is not changed during bending. Thus we are going to see the simple and pure bending only.

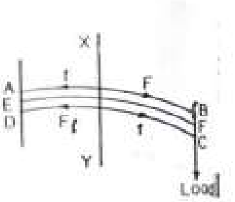
2.2 Plane of bending and Neutral axis of a bent beam:

A beam may be considered as consisting of a number of thin plane horizontal layers called **surfaces** placed one above the other. Now each plane layer or surface consists of a number of parallel longitudinal metallic fibres placed side by side and are called longitudinal filaments lying on convex side of the bent beam are elongated and those lying on concave side are shortened. However some of the filaments lying in the median plane of the beam remain unaltered in length and are called **neutral filaments** and the median plane containing these is known as **neutral surface**.



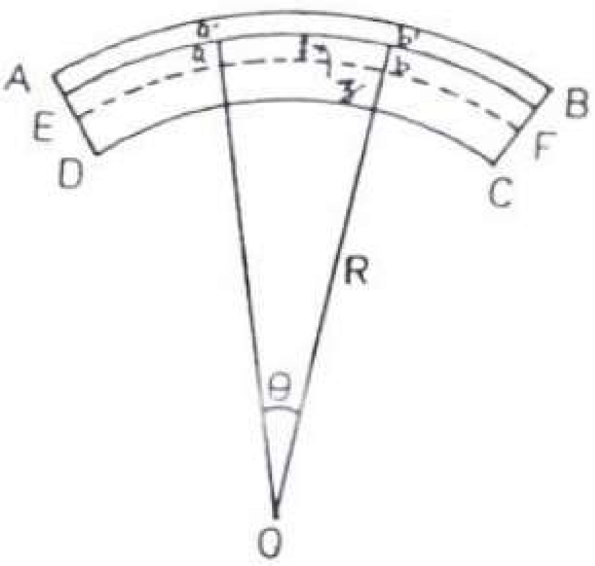
The plane in which bending takes place is known as **plane of bending** and obviously it is the vertical plane when the beam is placed horizontally. The line attained by the intersection of neutral surface and plane of bending is called **neutral axis**. A line perpendicular to the plane of bending on which centre of curvature of all the bent filaments lie is called **axis of bending.**

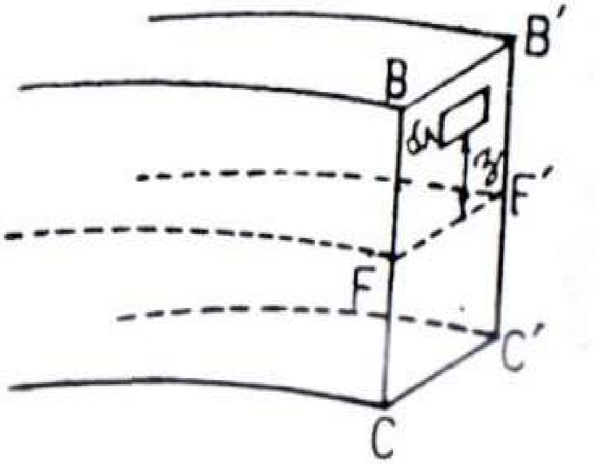
2.3 Bending moment:



In the bent beam let EF be the neutral surface (in the above fig). The plane XY represents a transverse section of the beam normal to EF. The filament AB, shown above the neutral surface gets elongated and thus it is under a stretching force F. Similarly the filament CD, shown below the neutral surface, gets compressed and thus it is under a compressive force 'F'. These two forces constitute a clockwise couple. This couple is called **external couple** or

bending couple and it has atendency to rotate the beam clockwise. Again as the beam is at rest, the moment of this couple must be balanced by an internal couple tending to rotate the beam anticlockwise. Thus when the filament above EF is stretched by F, an equal but opposite restoring force f arises in it. Similarly when the filament below EF is compressed by F, an equal and opposite restoring force f arises in it. These forces f and f constitute an anticlockwise internal couple which is called **balancing couple** or **restoring couple** which has a tendency to rotate the beam anticlockwise. If the moments of all external couples acting over all filaments in the cross section XY are added we get the moment of the external couple which bends the beam. If the moments of all the internal couples acting over all filaments in the section XY are added we get the moment of the internal restoring couple which balance the external couple. Thus in equilibrium position,





moment of bending couple = moment of restoring couple.

The moment of the internal restoring couple is called bending moment or internal bending moment of the beam

Let a beam ABCD having rectangular cross section be bent in the form of an arc of a circle of radius R with the centre at O. Consider a small portion ab of neutral axis of the beam subtending an angle θ at the centre O. a'b' is another small portion of a filament at a distance 'z' above the neutral filament ab. Before bending a'b' = ab. After bending, a'b' > ab since a'b' is above the neutral surface.

When, θ is small

$$a'b' = (R + z)\theta$$

$$ab = R\theta$$

Therefore increase in length of small element a'b' = a'b' – ab.

$$= (R + z)\theta - R\theta$$

$$= z\theta$$

Strain in a'b'= $\frac{increaseinlength}{originallength} = \frac{Z\theta}{R\theta} = \frac{Z}{R}$

Let BB′C′C be the cross section of the beam perpendicular to plane of bending (refer figure 2.). The line FF' lies in the neutral surface. Let us consider an area of cross section δA of a′b' at a distance z above the neutral line FF' on this cross section BB'C'C.

The Young's Modulus of the material of the beam 'E' = $\frac{Stress}{Strain}$

Therefore, stress on this area $\delta A = E.Strain = E\frac{Z}{R}$

Total internal force on the area

$$\delta A = E\frac{Z}{R}\delta A$$

Moment of this force about the neutral line FF'

$$= E\frac{z}{R}\delta A.z = \frac{E}{R}\delta Az^2$$

So the total moment of these internal forces acting above and below the neutral line

FF' = $\frac{E}{R}\sum \delta Az^2$

where $\sum \delta Az^2 = I_g$ the geometrical moment of inertia of the cross section area of the beam about a horizontal axis through its centroid. I_g is also equal to AK^2 where A is the cross sectional area of the beam and K, the radius of gyration of this cross sectional area about a horizontal axis through its centroid.

Thus the moment of the restoring .couple or the bending moment

$$= \frac{E}{R}I_g$$

As discussed above, $\frac{EIg}{R}$, the moment of all the internal forces balances the external couple.

The quantity $EI_g = EAK^2$ is called the flexural rigidity of the beam. Geometrical moment of inertia of the beam is also equal to the (mechanical) moment of inertia 'I' if the beam has an unit mass per unit area.

(i) If the cross section of the beam is rectangular then A = b x d where b is the breadth of the face BB'C'C and d the thickness of the beam. The moment of inertia of the rectangle BB'C'C about the axis FF' parallel to the side $BB' = MK^2 = M\frac{d^2}{12}$

(i.e) $K^2 = d^2/12$

Therefore, geometrical moment of inertia of the beam

$$I_g = AK^2 = \frac{bdd^2}{12} = \frac{bd^3}{12}$$

(ii) If the cross section of the beam is circular and has a radius 'r', then $A = \pi r^2$.

Moment of inertia about $FF^1 = MK^2 = M\frac{r^2}{4}$

Therefore, $K^2 = \frac{r^2}{4}$

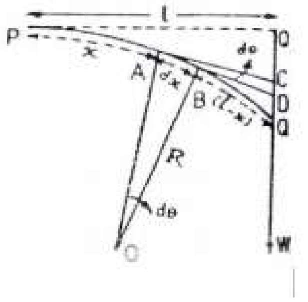
Hence geometrical moment of inertia of the beam about the neutral surface

$$I_g' = \pi r^2 \frac{r^2}{4} = \frac{\pi r^4}{4}$$

Depression of a cantilever:

A cantilever is a beam fixed horizontally at one end and loaded at the other end. The Young's modulus of the material of the cantilever can be determined using the value of depression produced in that cantilever.

Let PQ be the neutral axis of a cantilever fixed at P. Let ℓ be its length and the weight of the cantilever be negligible. It is loaded at Q with a weight and so the end Q is depressed to Q'.



Consider a point A at a distance x from the fixed end 'P' as shown in the above figure

The moment of the couple due to the load 'W' = bending couple

$$= W.AQ = W(\ell - x)$$

This must be equal to the moment of restoring couple (or) bending moment, $\frac{EI_g}{R}$, under equilibrium conditions.

$$\text{Therefore, } W(\ell - x) = \frac{EI_g}{R}$$

Where R is the radius of curvature of the neutral axis at A. Let B be another point at a distance dx from A and AB subtending an angle 'dθ' at O. When θ is small, dx = Rθ dx

$$\text{Hence, } R = \frac{dx}{D\theta}$$

Substituting the value of R in the above equation, we get

$$W(\ell - x) = EI_g \frac{d\theta}{dx}$$

$$W(\ell - x)dx = EI_g d\theta$$

Draw tangents at A and B meeting the vertical line OQ' at C and D respectively.

Then the depression of B below A is evidently

$$CD = dy = (\ell - x)d\theta$$

$$\left(\frac{dy}{\ell - x}\right) = d\theta$$

Substituting the value of $d\theta$ in the above equation, we get

$$W(\ell - x)dx = EI_g\left(\frac{dy}{\ell - x}\right)$$

$$i.e. W(\ell - x)^2 dx = EI_g dy$$

$$\text{Thus } dy = \frac{W(\ell - x)^2}{EI_g}dx$$

Therefore total depression of the cantilever 'y'

$$= \frac{W}{EI_g}\int_0^{\ell}(\ell - x)^2\ dx = \frac{W}{EI_g}\int_0^{\ell}(\ell^2\ - 2\ell x + x^2)\,dx$$

$$= \frac{W}{EI_g}\left[\ell^2\,x - \frac{2\ell\ \ x^2}{2} + \frac{x^3}{3}\right]_0^{\ell} = \frac{W}{EI_g}\left(\ell^3 - \ \ell^3 + \frac{\ell^3}{3}\right)$$

$$y = \frac{W\ell^3}{3EI_g}$$

Hence the Young's modulus of the material of the cantilever

$$E = \frac{W\ell^3}{3yI_g}$$

Note: When the weight of the cantilever is effective, then in addition to the weight W at Q the weight of the portion $(\ell - x)$ of the cantilever is also acting at the mid point or the centre of gravity of this portion. If w be the weight per unit length of the cantilever, a weight of

w $(\ell - x)$ is acting at a distance $(\ell - x)/2$ from the section AB.

Under equilibrium conditions, the total bending moment,

$$W(\ell - x) + w(\ell - x)\frac{(\ell - x)}{2} = \frac{EI_g}{R} = EI_g \frac{d\theta}{dx}$$

$$\frac{w(\ell - x)dx + \frac{w}{2}(\ell - x)^2 dx}{EI_g} = d\theta$$

Therefore dy = $(\ell - x)d\theta = \frac{W}{EI_g}(\ell - x)dx + \frac{w}{2EI_g}(\ell - x)^3\, dx$

$$\therefore Total\ deepression\quad 'y' = \frac{W}{EI_g}\int_o^{\ell}(\ell - x)^2\, dx + \frac{w}{2EI_g}\int_o^{\ell}(\ell - x)^3\, dx$$

Put $(\ell - x) = u$

Therefore $-dx = du$ and $\int_o^{\ell} dx = \int_o^{\ell} du$

Hence $y = \frac{W}{EI_g}\int_o^{\ell} u^2 du + \frac{w}{2EI_g}\int_o^{\ell} u^3 du$

$$= \frac{W\ell^3}{3EI_g} + \frac{w\ell^4}{8EI_g}$$

If $W_1 = w\ell = weight\ of\ the\ cantilever, then$

$$y = \left(W + \frac{3}{8}W_1\right)\frac{\ell^3}{3EI_g}$$

When $W_1 \ll W \quad then\ y = \frac{W\ell^3}{3EI_g}$

SURFACE TENSIO

Surface tension is that property of liquids owing to which they tend to acquire minimum surface area.

Small liquid drops acquire spherical shape due to surface tension. Big drops flatten due to weight.

The following experiment illustrates the tendency of a liquid to decrease its surface area.

When a camel hair brush is dipped into water, the bristles spread out [Fig. 3.1 (a)]. When the brush is taken out, the bristles cling together on account of the films of water between them contracts [Fig. 3.1 (b)]. This experiment clearly shows that the surface of a liquid behaves like an elastic membrane under tension with a tendency to contract. This tension or pull in the surface of a liquid is called its surface tension.

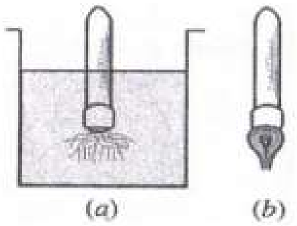


Fig. 3.1

Definition: Surface tension is defined as the force per unit length of a line drawn in the liquid surface, acting perpendicular to it at every point and tending to pull the surface apart along the line.

: Surface tension is the ratio of a force to a length.=F/L

3.4 Excess pressure Inside a Liquid Drop

A spherical liquid drop has a convex surface [Fig. 3.4 (i)]. The molecules near the surface of the drop experience a resultant force, acting inwards due to surface tension. Therefore the pressure inside the drop must be greater than the pressure outside it. Let this excess pressure inside the liquid drop over the pressure outside it he *p*.

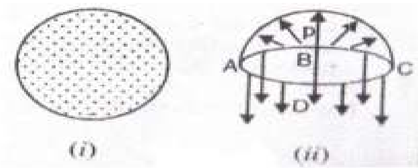


Fig.3.4

Imagine the drop to be divided into two exactly equal halves. Consider the equilibrium of the upper half of the drop [Fig. 3.4 (ii)]. r is the radius of the drop and σ it's Surface Tension.

Upward force on the plane face ABCD due to the excess pressure p $= p \pi r^2$

Downward force due to surface tension acting along the circumference of the circle ABCD $= \sigma 2 \pi r$

$$p \pi r^2 = \sigma 2 \pi r$$
$$p = \frac{2\sigma}{r}$$

3.5 Excess Pressure inside a Soap Bubble

Consider a soap bubble of radius r [Fig. 3.5 (i)]. Let p be the excess pressure inside it. A soap bubble has two liquid surfaces in contact with air, one inside the bubble and the other outside the bubble. σ is the surface tension of soap solution.

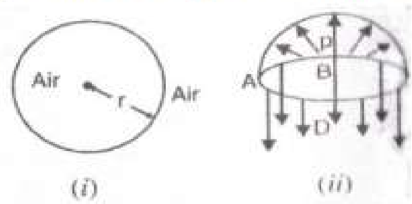


Fig.3.5

Consider the equilibrium of the upper half (or the upper hemisphere of the bubble [Fig. 3.5 (ii)].

Upward force on the plane face ABCD due to the excess pressure p $= p \pi r^2$

Downward force due to surface tension acting along the circumference of the circle ABCD $= 2 \times \sigma 2 \pi r = 4 \pi r \sigma$

For equilibrium of the hemisphere,

$$p \pi r^2 = 4 \pi r \sigma$$
$$p = \frac{4\sigma}{r}$$

Fluid Motion

Consider two parallel planes A and B separated by a distance [Fig.4.1]. Let the plane B be at rest and the plane A be moving with a uniform velocity. Let the space between the two plans be filled with a gas or liquid. So the layer of liquid in contact with B will be a rest while the layer in contact with A will move the maximum velocity. The layers in between will move the different velocities, decreasing from A to B. So a velocity gradient is set up. Because of this, the liquid will exert a force at A in a direction opposite to its motion tending to reduce its velocity. Similarly at B, a force will be exerted urging it to move in the direction of motion of A. As a net result, the relative velocity between the layers A and B will gradually decreases. This property of the liquid by which it resists the relative motion between its different layers is known as viscosity or internal friction of the liquid.

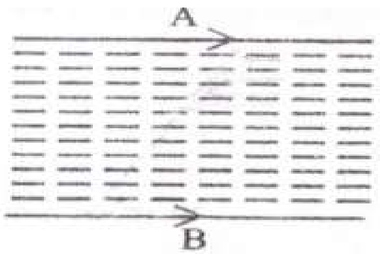


Fig. 4.1

It is to be noted that an external tangential force has to be applied to maintain this relative motion between the layers of the liquid; otherwise the liquid will not flow. According to Newton's third law of motion, internal forces from within the liquid will be brought into play opposing the flow of the liquid. These internal forces are called viscous forces or viscous drag. These forces are similar to forces of friction in solids.

4.2 Stream - lined, and turbulent motion

Consider the flow of liquid through a narrow tube. The velocity of flow is greatest along the axis of the tube. The layers in contact with the walls of the tube will be at rest. An external pressure - head is to be applied for the liquid to flow. If this pressure - head is constant, the liquid settles down into steady motion. Each particle will move parallel to the axis of the liquid, with a constant velocity gradient along the radius of the tube. This orderly motion is possible only if the tube is narrow enough and the pressure head is not large. Such a smooth and orderly motion of the liquid is called **stream - lined motion.**

If however, the pressure - head is large enough, the liquid particles are accelerated axially. Now the resultant motion of the liquid is not orderly, but violent. Such a motion is called **turbulent motion**.

The velocity of the fluid at which orderly motion ceases and **turbulent** motion sets in is known as the **Critical Velocity** (V_c) For orderly motion, external pressure head $P \alpha V_c^2$. It has been found that

$$V_c = \frac{K\eta}{\rho r}$$

Where ρ is the density of the liquid, η is the viscosity of the liquid and r is the radius of the tube. K is a constant called Reynolds' number. For narrow tubes, K is approximately 1000.

4.3 Coefficient of Viscosity

Consider a liquid flowing over a horizontal surface. The layer in contact with the surface is at rest. The velocities of other layers increase uniformly from layer to layer. The velocity is maximum for the top layer [Fig. 4.2].

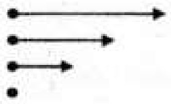


Fig. 4.2

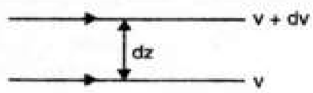


Fig. 4.3

Consider two layers of liquid separated by a distance dz [Fig. 4.3]. Let υ and $\upsilon + d\upsilon$ be the velocities of two layers. So the velocity gradient is $d\upsilon/dz$. Let A be surface area of the layer. The viscous force is directly proportional to the surface area A and velocity gradient $d\upsilon/dz$.

$$F = A\frac{d\upsilon}{dz} \text{ or } F = \eta\, A\frac{d\upsilon}{dz}$$

Definition: The coefficient of viscosity is defined as the tangential force per unit area required to maintain a unit velocity gradient.

4.4 Rate of Flow of Liquid in a Capillary Tube - Poiseuille's formula

Consider horizontal capillary tube to length l and radius a through which a liquid flows [Fig. 4.4] is the coefficient of viscosity of the liquid. p is the pressure difference between the ends of the tube. The velocity of the liquid is maximum along and is zero at the walls. (dv/dr) is the velocity gradient.

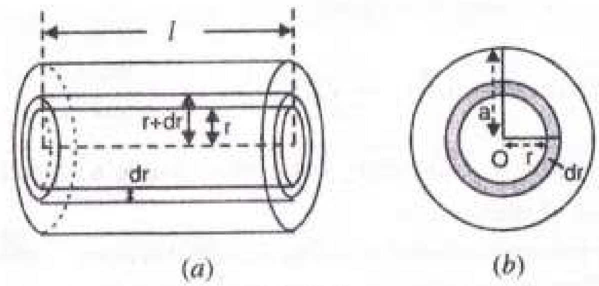


Fig. 4.4

Consider a cylindrical shell of the liquid of inner radius r an outer radius r+dr (Fig. 4.4 (b)]

The surface area of the shell = A = $2\pi rl$.

The backward dragging viscous force acting on this layer is

$$F_1 = -\eta A\frac{dv}{dr} = -\eta 2\pi rl\frac{dv}{dr}$$

The driving force on the liquid shell, accelerating it forward

$$F_2 = px\pi r^2$$

$\pi r^2 = Area\ of\ cross -$ section of the innercylinder.

Here,

When the motion is steady,

backward dragging force (F_{1}) = driving force (F_2)

$$-\eta 2\pi r l \frac{dv}{dr} = p\pi r^2$$

$$dv = \frac{-p}{2\eta l} r\, dr$$

Integrating, $v = \frac{-p}{2\eta l}\frac{r^2}{2} + C$ (C is constant)

When $r = a, v = 0$

$$0 = \frac{-p}{2\eta l}\frac{a^2}{2} + C.$$

or $C = \frac{pa^2}{4\eta l}$

$$v = \frac{p}{4\eta l}(a^{2-r^2}).$$

Volume of liquid flowing per second through this shell

dV = (Area of cross –section of the shell) x Velocity

$=[2\pi\, r\, dr]\ \left[\frac{p}{4\eta l}(a^2 - r^2\right]$

$\frac{\pi p}{2\eta l}(a^2 r - r^{3)}\, dr$

The Volume of the liquid flowing out per second is obtained by integrating the expression for dv between the limits r = 0 to r=a

$$V = \int_0^a \frac{\pi p}{2\eta l}(a^2 r - r^3)dr = \frac{\pi p}{2\eta l}\left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^a$$

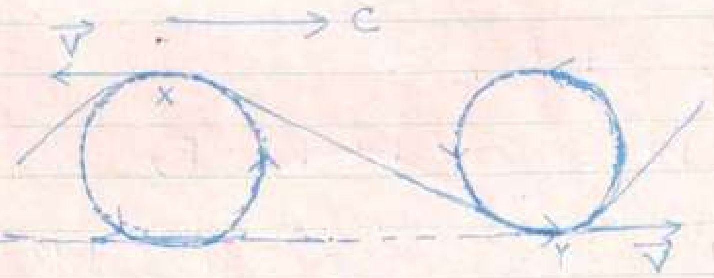
$$= \frac{\pi\, p}{2\eta l}\frac{a^4}{4}$$

$$V = \frac{\pi p a^4}{8\eta l}$$

This is Poiseuille's formula for the rate of flow of liquid through a capillary tube.

Ripples and Gravity Waves:

When a wave propagates through a deep liquid, the particles of the liquid describe circular motion. The radius of the circle decreases as the depth increases.



Let us consider a wave propagating through a liquid.

$\vec{C}$ = The velocity of propagation of the wave

λ = wave length of the wave.

r = Radius of the circle; described by the particles of the liquid.

ω = angular velocity of rotation of the particles.

V = Linear velocity of the liquid particles describing circular motion = $r\omega$

The motion appears to be stable to an observer moving with the velocity $\vec{C}$ along the direction of propagation of the wave.

But to an Observer at rest the actual motion is the resultant of a steady velocity $\vec{c}$ & constant speed v.

Let us consider two liquid particles at X & Y, the highest and the lowest point of the circle respectively.

Let q_1 and q_2 be the velocity at points X & Y respectively.

$$\boxed{q_1 = c - v} \quad \& \quad \boxed{q_2 = c + v}$$

Let the lowest point, be taken as reference level.

$$E_x = g \cdot 2r + \frac{1}{2} \cdot a_1^2$$

$$E_y = 0 + \frac{1}{2} a_2^2$$

Since energy is conserved : $E_x = E_y$

$$g \cdot 2r + \frac{1}{2} a_1^2 = \frac{1}{2} a_2^2 \quad \text{or} \quad a_2^2 - a_1^2 = 4gr$$

$$(a_2 + a_1)(a_2 - a_1) = 4gr$$

$$(c + v + c - v)(c + v - c + v) = 4gr$$

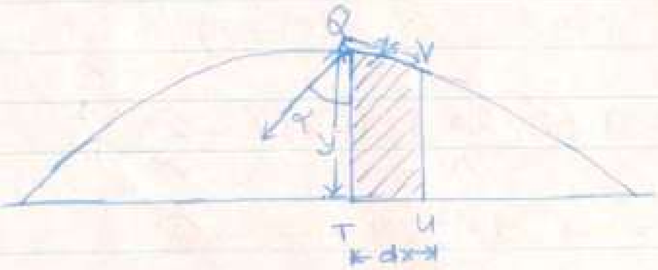
$$2c \cdot 2v = 4gr \quad \text{or} \quad c\,\omega r = g r$$

$$\text{or} \quad c \cdot \frac{2\pi c}{\lambda} = g. \qquad \left[\because \omega = 2\pi n ; n = \frac{c}{\lambda}\right]$$

$$\text{or} \quad c^2 = \frac{g\lambda}{2\pi} \quad \text{or} \quad c = \sqrt{\frac{g\lambda}{2\pi}} \longrightarrow (1)$$

Equation (1) gives the velocity of propagation of a wave through a liquid, under the effect of gravity alone.

We now consider; the effect of surface Tension on the velocity of propagation of wave through a liquid.
let us consider a plane wave, propagating through a liquid. In the fig. a section of the plane wave by a vertical plane is shown



Given: S: Surface tension of the liquid.
ρ = density of the liquid.

Let us consider an element of the wave QTUV of height 'y' width 'dx' and arc tual length ds
Let 'r' & R be the principal radii of curvature at the point Q.
Excess pressure at Q: $P = S\left(\frac{1}{r} + \frac{1}{R}\right)$

Since for a plane wave; one of the principal radii of curvature 'R' is infinity ($R = \infty$)
$\frac{1}{R} = 0$
Excess pressure at Q; $P = \frac{S}{r} \rightarrow$ (I)

Excess pressure: $P = \frac{S}{r}$ = +ve on the Concave side
− ve on the Convex side (I)
Using the relation from Calculus:-
$\frac{1}{r} = \frac{d^2y}{dx^2}$ $\therefore$ $P = S.\frac{d^2y}{dx^2} \rightarrow (1a)$

We know that the equation of a plane progressive wave is

$$y = a\sin\frac{2\pi}{\lambda}(ct - x) \longrightarrow ②$$

Differentiating equation ② w.r.t. to x:-

$$\frac{dy}{dx} = a\cos\frac{2\pi}{\lambda}(ct-x)\left(-\frac{2\pi}{\lambda}\right)$$

or $$\frac{d^2y}{dx^2} = -a\sin\frac{2\pi}{\lambda}(ct-x)\left(-\frac{2\pi}{\lambda}\right)^2$$

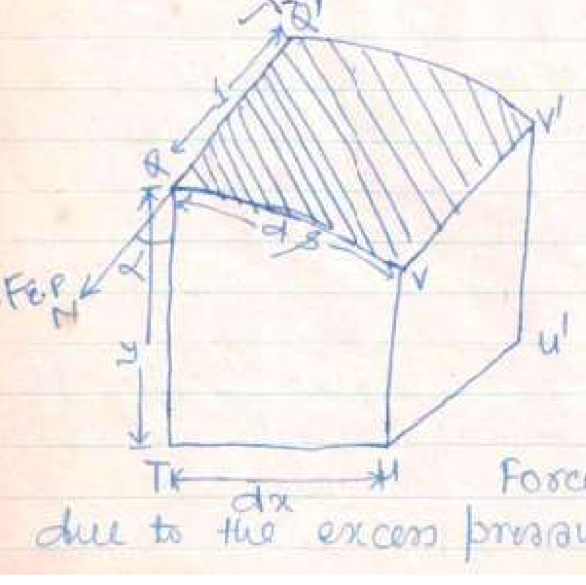
or $$\frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2}.y \longrightarrow ③$$

Putting equation ③ in (1a)

Excess pressure $P = S\left(-\frac{4\pi^2}{\lambda^2}y\right)$ = Pre. on concave side - pre. on convex side

The (-)ve sign indicates that the pressure on the convex side is greater than the pressure on concave side.

$$+\frac{S.4\pi^2}{\lambda^2}.y = \text{Pre. on the convex} - \text{Pre. on the concave} \longrightarrow ④.$$



let us consider an element of the wave of height y, breadth 'dx' width 'unit'

Area of the curved surface $= (ds \times 1)$

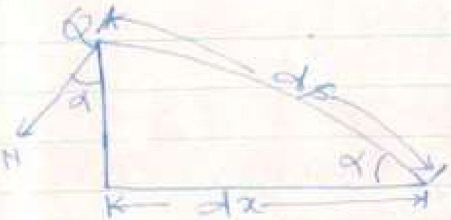
Force on the liquid element due to the excess pressure is along the normal $\overrightarrow{QN}$

to the curved surface $= P \times (ds \times 1) = \frac{4\pi^2}{\lambda^2} S y ds$.

Let the normal $\vec{QN}$ makes an angle 'α' with the vertical.

The component of the force due to the excess pressure i.e the force due to the S.T along vertically downward direction.

$$= \frac{4\pi^2}{\lambda^2} S y \, ds \cos\alpha \longrightarrow (5)$$



From the fig; $\cos\alpha = \frac{dx}{ds}$

or $ds \cos\alpha = dx \longrightarrow (6)$

Putting equation (6) in (5); the force on the liquid element; due to S.T along vertically downward direction

$$= \frac{4\pi^2}{\lambda^2} S y \, dx \longrightarrow (7)$$

The force on the liquid element; due to gravity along vertically downward direction

Side. $= (y \times dx \times 1) \rho g \dashrightarrow (8)$.

Hence the total force on the liquid element along vertically downward direction; due to gravity as well as S.T

$$= y\rho g \, dx + \frac{4\pi^2}{\lambda^2} S y \, dx$$

$$= y\rho dx \left[g + \frac{4\pi^2}{\lambda^2} \frac{S}{\rho} \right] \longrightarrow (9)$$

$$= y\rho dx \, g'$$

where $g' = g + \frac{4\pi^2 S}{\rho \lambda^2} \longrightarrow (10)$.

Thus from (10) we can conclude that; if we consider the effect of S.T along with gravity 'g' modifies to g' hence the velocity of propagation of wave through a liquid under the effect of gravity and S.T both; can be obtained on replacing 'g' in equation (1) by 'g''

i.e $C = \sqrt{\frac{g'\lambda}{2\pi}} = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi s}{\rho\lambda}} \rightarrow$

$$\because g' = g + \frac{4\pi^2 s}{\rho\lambda^2}$$

From (11) we find that
When λ decreases, C increases.
when λ increases, C increases.

- Hence there must be a particular value of λ for which 'c' is minimum, that value of λ is known as λ_m, which can be found using the results from calculus $\frac{dc}{d\lambda} = 0$

$$\frac{dc}{d\lambda} = \frac{1}{2}\left[\frac{g\lambda}{2\pi} + \frac{2\pi s}{\rho\lambda}\right]^{-\frac{1}{2}}\left[\frac{g.1}{2\pi} + \frac{2\pi s}{\rho}\left(-\frac{1}{\lambda^2}\right)\right]$$

g, λ, S, ρ, π all being +ve; the 1st bracketed term being sum of two +ve quantities cannot be zero.

$$\frac{g}{2\pi} - \frac{2\pi s}{\rho\lambda^2} = 0$$

$$\frac{2\pi s}{\rho\lambda_m^2} = \frac{g}{2\pi}$$

$$\lambda_m^2 = \frac{4\pi^2 s}{\rho g}$$

$$\lambda_m = 2\pi\sqrt{\frac{s}{\rho g}} = \text{Const} \longrightarrow (12)$$

Thus for a given liquid; there is a particular Const. wave length given by equation (12) for which the velocity of the wave is minimum.

Case I: If the wave which propagates through the liquid has wave length λ greater than this value λ_m i.e $\lambda \gg \lambda_m$ from (11) $\frac{g\lambda}{2\pi} \gg \frac{2\pi S}{\rho\lambda}$

Hence the effect of gravity is more important than the effect of surface tension and the wave is known as Gravity wave.

Case II: $\lambda < \lambda_m$ $\frac{2\pi S}{\rho\lambda} \gg \frac{g\lambda}{2\pi}$, the effect of S.T is more important than the effect of gravity and the wave is known as Ripple.

$$C = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi S}{\rho\lambda}} \qquad \boxed{C = n\lambda}$$

$$n^2\lambda^2 = \frac{g\lambda}{2\pi} + \frac{2\pi S}{\rho\lambda}$$

or $\frac{2\pi S}{\rho\lambda} = n^2\lambda^2 - \frac{g\lambda}{2\pi}$

$$S = \frac{n^2\lambda^3\rho}{2\pi} - \frac{g\rho\lambda^2}{4\pi^2} \longrightarrow (13)$$

Using equation (13), surface tension of the liquid can be experimentally determined.

Discussions: $W = mg$ or $g = \frac{W}{m}$ = The gravitational force per unit mass hence following the analogy the term $\frac{4\pi^2 S}{\rho\lambda^2}$ can be called

Case III: For $\lambda = 1.7$ cm from Ratio $\simeq 1$ i.e both surface tension and gravity are equally important. Hence for experimental determination of S.T of a liquid, by this method we must prefer waves having wave length around 1.7 cm.