3RD SEM
(Paper-V)

LESSON PLAN

Class : 3rd sem Sub : No. of Periods / Weeks :

Sl.No.	Month	Paper/ Unit	Topics assigned	Page No.
1	2	3	4	5
	NOVEMBER	3rd unit II	frobenius method and special function:- singular points of second order linear Differential Equation and their importance, frobenius method and its application to differential Equations. legendre and Hermite Differential Equation, properties of legender and Hermite Polynomials. Rodrigues formula.	

PROGRESS

Sl.No.	Date	Time	Topics covered (If class not taken, mention the reasons)	Signature of Teacher
1	2	3	4	5
			9-11 - Exam	
	9.11.20	10-11	Introduction for self study	[signature]
	10.11.20	10-11	Singular points of second order linear Differential equation	[signature]
	11.11.20	10-11	Frobenius method and its application to differential equations	[signature]
	14.11.20	11-12	Brimage dissection and Hadding	
	20.11.	10-11	Legendre and Hermite. Differential equation.	[signature]
	21.11		Revision	
	23.11	10-11	properties of legendre and Hermite polynomials.	[signature]
			Revision	[signature]
	25.11	10-11	Generating function	[signature]
	26.11	10-11	- Doubt clearing with some example solution	[signature]
	27.11.	10-11	Orthogonality	

[signature] 30/11/2020

CHAPTER 3

FROBENIUS METHOD & SPECIAL FUNCTIONS

LEARNING OBJECTIVES

- Frobenius method, Frobenius method.
- Frobenius Method of series solution when the Two Roots of Indicial Equation are Equal.
- Legenre's differential equation, Generating Function for $P_n(x)$, Rodrigue's odrigue's formula, Legendre's polynomials.
- Recurrence formula for legendre's polynomials, orthogonal property of legendre's polynomial.
- Expansion of Function f(x) in a Series of Legendre Polynomials, Hermite's differential equation, Hermite Polynomials, Generating Function for Hermite Polynomials.
- Rodrigue's Formula for Hermite Polynomials, Orthogonality of Hermite Polynomials. Recurrence Formula for Hermite Polynomials, Associated legendre polynomials
- Recurrence Relations, Orthogonality Relation for $P_n^m(x)$, Spherical harmonics.

FROBENIUS METHOD : (For Power Series Solutions of Differential Equations)

A homogeneous linear second order differential equation with variable coefficients is given by

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \qquad \text{........... (1)}$$

∴ Dividing equation (1) by $a_0(x)$, we get :

$$\frac{d^2y}{dx^2}+\frac{a_1(x)}{a_0(x)}\frac{dy}{dx}+\frac{a_2(x)}{a_0(x)}y=0 \qquad \text{............ (2)}$$

Writing $P(x)=\frac{a_1(x)}{a_0(x)}$ and $Q(x)=\frac{a_2(x)}{a_0(x)}$

Equation (2) becomes $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$ .... (3)

which is the standard or Normal from.

Ordinary Point : A point x_0 is said to be an ordinary point or Regular point of the differential Equation (3) if both P(x) and Q(x) are analytic at x_0.

A function f(x) is said to be analytic at x_0 if f(x) can be expand in Taylor's expansion about x_0 given by $\sum_{n=0}^{\infty}\frac{f^n(x_0)}{n!}(x-x_0)^n$ which exist and converges to f(x).

Hence x_0 is an ordinary point if $a_0(x_0) \neq 0$ i.e $P(x) \neq \infty$, $Q(x) \neq \infty$.

Singular Point : A point x_0 is said to be singular point of equation (3) if either P(x) or Q(x) or both are not Analytic at x_0.

So at $x = x_0$, $P(x) = \infty$ or $Q(x) = \infty$.

Singular point is of two types.

(a) Regular Singular point :

The singular point x_0 of differential equation (2) is know as a regular singular point if both $(x - x_0)$ P(x) and $(x - x_0)^2Q(x)$ are analytic at $x = x_0$. In other words if $(x - x_0)p(x)$ and $(x - x_0)^2Q(x)$ are not infinite at $x = x_0$ then $x = x_0$ is a regular singular point. In this case equation (2) takes the form $\frac{d^2y}{dx^2}+\frac{P(x)}{(x-x_0)}\frac{dy}{dx}+\frac{Q(x)y}{(x-x_0)^2}=0.$

(b) Irregular Singular point :

The point $x = x_0$ is called an Irregular singular point of equation (2) if $(x - x_0)P(x)$ or $(x - x_0)^2Q(x)$ or both are not analytic at x_0.

So $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are infinite at $x = x_0$. Then x_0 is Irregular singular pint.

Find the regular and Singular point of the differential Equation below :

Example - 1 : $\frac{d^2y}{dx^2}+\left(x^3+x^2+1\right)\frac{dy}{dx}-3\left(x^2-4x-2\right)y=0.$

Solution. Comparing with standard or Normal form of the differential Equation we get :

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0,$$

We get $P = x^3 + y^2 + 1$, $Q = -3(x^2 - 4x - 2)$ which are polynomials and both are analytic everywhere. Hence any point is an ordinary or Regular point of the differential equation.

Example - 2 : $(1 - x^2)y'' - 2xy' + m(m + 1)y = 0$.

Solution. $(1 - x^2)y'' - 2xy' + m(m + 1)y = 0$

$$\Rightarrow \frac{d^2y}{dx^2}-\frac{2x}{\left(1-x^2\right)}\frac{dy}{dx}+\frac{m(m+1)y}{\left(1-x^2\right)}=0$$

Comparing with $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$, we get $P(x)=\frac{-2x}{1-x^2}, Q(x)=\frac{m(m+1)}{1-x^2}$

which are not analytic at $x = \pm 1$ ($P \to \infty$, $Q \to \infty$ when $1 - x^2 = 0$ or $x^2 = 1 \Rightarrow x = \pm 1$)

So $x = \pm 1$ are singular points of equation.

Example - 3 : $x^2y'' + xy' + by = 0$.

Solution. $x^2y'' + xy' + by = 0$

$$\Rightarrow y'' + \frac{y'}{x} + \frac{b}{x^2}y = 0$$

$\therefore P = \frac{1}{x}$ and $Q = \frac{b}{x^2}$ both become infinite when $x = 0$

So $x = 0$ is the singular point of the equation.

Example - 4 : $x^2y'' + xy' + (x^2 - n^2)y = 0$.

Solution. $x^2y'' + xy' + (x^2 - n^2)y = 0$

$$\Rightarrow y'' + \frac{y'}{x} + \left(\frac{x^2 - n^2}{x^2}\right)y = 0$$

Then $P(x) = \frac{1}{x}, Q(x) = \frac{x^2 - n^2}{x^2}$

$\therefore$ both $P(x) \to \infty$ when $x = 0$
$Q(x) \to \infty$

So $x = 0$ is singular point of the equation.

Example - 5 : $x(x - 2)^2y'' + 2(x - 2)y' + (x + 3)y = 0$

Solution. $x(x - 2)^2y'' + 2(x - 2)y' + (x + 3)y = 0$

$$\Rightarrow y'' + \frac{2(x-2)}{x(x-2)^2}y' + \frac{(x+3)}{x(x-2)^2}y = 0$$

$$\Rightarrow y'' + \frac{2}{x(x-2)}y' + \frac{x+3}{x(x-2)^2}y = 0$$

Hence we get : $P(x) = \frac{2}{x(x-2)}, Q(x) = \frac{x+3}{x(x-2)^2}$

Hence At $x = 0$ or $x = 2$ both $P(x)$ and $Q(x)$ are ∞.

i.e $P(x)$ and $Q(x)$ are not analytic at $x = 0$, and $x = 2$

Hence 0 and 2 are singular points of the equation.

Find the type of singular points of the following equations.

Example - 1 : $(1 - x^2)y'' - 2xy' + m(m + 1)y = 0$.

Solution. $(1 - x^2)y'' - 2xy' + m(m + 1)y = 0$

i.e $y'' - \left(\frac{2x}{1-x^2}\right)y' + \frac{m(m+1)}{(1-x^2)}y = 0$

$\therefore P(x) = \frac{-2x}{1-x^2}, Q(x) = \frac{m(m+1)}{1-x^2}$ As already discussed above $x = \pm 1$ are singular points of the equation.

$\therefore (x-1)P(x) = (x-1)\left(\frac{-2x}{1-x^2}\right)$

$= \frac{(1-x)}{(1+x)}\left(\frac{2x}{1-x}\right) = \frac{2x}{1+x}$ is analytic at $x = 1$.

Also $(x-1)^2 Q(x) = (x-1)^2 \frac{m(m+1)}{(1-x^2)} = \frac{(x-1)(x-1)m(m+1)}{(1-x)(1+x)} = \frac{-(x-1)m(m+1)}{(1+x)}$

$= \frac{(1-x)m(m+1)}{1+x}$ are analytic at $x = 1$.

So $x = 1$ is a regular singular point.

Similarly both $(x + 1)P(x)$ and $(x + 1)^2Q(x)$ are analytic at $x = -1$.

$\{(x + 1)P(x) \neq \infty, (x + 1)^2Q(x) \neq \infty$ at $x = -1\}$

Therefore $x = -1$ is also regular singular point.

Example - 2 : $x^3(x - 3)y'' + 2(x - 3)y' + 6xy = 0$.

Solution. $x^3(x - 3)y'' + 2(x - 3)y' + 6xy = 0$

$$\Rightarrow y'' + \frac{2(x-3)}{x^3(x-3)}y' + \frac{6xy}{x^3(x-3)} = 0$$

$$\Rightarrow y'' + \frac{2}{x^3}y' + \frac{6y}{x^2(x-3)} = 0$$

Here $P(x) = \frac{2}{x^3}, Q(x) = \frac{6}{x^2(x-3)}$

$\therefore x = 0$ and $x = 3$ are singular points

Now $xP(x) = x\frac{2}{x^3} = \frac{2}{x^2} = \infty$ as $x = 0$

So $x = 0$ is an Irregular singular point.

Now $(x-3)P(x) = (x-3)\frac{2(x-3)}{x^3(x-3)} \neq \infty$ when $x = 3$

Also $(x-3)^2 Q(x) = (x-3)^2 \frac{6}{x^2(x-3)} = \frac{6(x-3)}{x} \neq \infty$ when $x = 3$

Then $x = 3$ is a regular singular point of the equation.

Example - 3 : $x^2(x^2-9)y'' + 2x^3y' + 4y = 0$.

Solution. $x^2(x^2-9)y'' + 2x^3y' + 4y = 0$

$$\Rightarrow y'' + \frac{2x^3y'}{x^2(x^2-9)} + \frac{4y}{x^2(x^2-9)} = 0$$

$$\Rightarrow y'' + \frac{2y'x}{(x^2-9)} + \frac{4y}{x^2(x^2-9)} = 0$$

$$\therefore P(x) = \frac{2x}{(x^2-9)},\ Q(x) = \frac{4}{x^2(x^2-9)}$$

$\therefore$ $x = 0$ and $x = \pm 3$ are singular points.

Because both P(x) and Q(x) are not analytic at $x = 0$, $x = \pm 3$

Now $xP(x) = x\frac{2x}{(x^2-9)} \neq \infty$ at $x = 0$, $x^2Q(x) = x^2\frac{4}{x^2(x^2-9)} = \frac{4}{x^2-9} \neq \infty$ at $x = 0$

$\therefore$ $x = 0$ is a Regular singular point

$$\therefore (x-3)P(x) = (x-3)\frac{2x}{(x^2-9)} = \frac{(2x)}{(x+3)} \neq \infty \text{ at } x = 3$$

$$(x-3)^2Q(x) = (x-3)^2\frac{4}{x^2(x^2-9)} = \frac{(x-3)(x-3)4}{x^2(x-3)(x+3)} = \frac{4(x-3)}{x^2(x+3)} \neq \infty \text{ at } x = 3$$

$\therefore$ $x = 3$ is also a regular singular point.

Similarly it can be shown that $x = -3$ is also a regular singular point of the given equation.

Example - 4 : $x^2y'' + (\sin x)y' + (\cos x)y = 0$.

Solution. $x^2y'' + (\sin x)y' + (\cos x)y = 0$

$$\Rightarrow y'' + \left(\frac{\sin x}{x^2}\right)y' + \left(\frac{\cos x}{x^2}\right)y = 0$$

Hence in the equation $P(x) = \frac{\sin x}{x^2}$, $Q(x) = \frac{\cos x}{x^2}$

Both P(x) and Q(x) are not analytic at $x = 0$

Hence $x = 0$ is singular point of the equation

But $xP(x) = x\cdot\frac{\sin x}{x^2} = \frac{\sin x}{x} \neq \infty$ as $x = 0$ $\left\{\sin x = \frac{x - x^3}{3!} + \frac{x^5}{5!} - \dots\dots\right.$

$$x^2Q(x) = x^2\frac{\cos x}{x^2} \neq \infty \text{ as } x = 0$$

Hence $x = 0$ is a regular singular point.

mple - 5 : $(x^2 + x - 2)^2y'' + 3(x + 2)y' + (x - 1)y = 0$.

Solution. $(x^2 + x - 2)^2y'' + 3(x + 2)y' + (x - 1)y = 0$

$$\Rightarrow y'' + \frac{3(x+2)}{(x^2+x-2)}y' + \frac{(x-1)}{(x^2+x-2)^2}y = 0$$

$$\Rightarrow y'' + \frac{3(x+2)}{(x-1)^2(x+2)^2}y' + \frac{(x-1)}{(x-1)^2(x+2)^2}y = 0$$

$$\Rightarrow y'' + \frac{3y'}{(x+2)(x-1)^2} + \frac{y}{(x-1)(x+2)^2} = 0$$

Then $P(x) = \frac{3}{(x+2)(x-1)^2}$, $Q(x) = \frac{1}{(x-1)(x+2)^2}$

Both P(x) and Q(x) are not analytic at $x = 1$, $x = -2$

Hence $x = 1$, and $x = -2$ are singular points

Now $(x-1)P(x) = (x-1)\frac{3}{(x+2)(x-1)^2} = \frac{3}{(x+2)(x-1)} = \infty$ at $x = 1$

Hence $x = 1$ is an Irregular singular point of the equation.

Also $(x+2)P(x) = (x+2)\frac{3}{(x+2)(x-1)^2} = \frac{3}{(x-1)^2} \neq \infty$ at $= -2$

$$(x+2)^2 Q(x) = (x+2)^2 \frac{1}{(x-1)(x+2)^2} = \frac{1}{(x-1)} \neq \infty \text{ at } x = -2$$

Hence both P(x) and Q(x) are analytic at $x = -2$.

Hence $x = -2$ is a regular singular point of the equation.

FROBENIUS METHOD

Power series solution at regular singular point x_0 can be obtained by frobenius method. If x_0 is the Regular singular point of the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \qquad \text{.............. (1)}$$

The series solution of equation (1) is in the form of

$$y = (x - x_0)^r[a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \text{..........}]$$

or $$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} \qquad \text{..........(2)} \; C_0 \neq 0$$

Differentiating equation (2) with respect to x :

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(k + r)a_k (x - x_0)^{k+r-1} \qquad \text{.............. (3)}$$

Also $$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n (x - x_0)^{k+r-2} \qquad \text{.............. (4)}$$

Putting these values in equation (1) and we collect the coefficients of like power of $(x - x_0)$ to get equation of the form

$$A_0(x - x_0)^{r+k} + A_1(x - x_0)^{r+k+1} + A_2(x - x_0)^{r+k+2} + \text{..........} = 0$$

where k is an integer.

As equation (2) in solution of equation (1), we have

$$A_0 = A_1 = A_2 = \text{..........} = 0$$

Here A_0 is the coefficient of lowest power $(r + k)$ of $(x - x_0)$

Then $A_0 = 0$ is quadratic in r and is called Indicial equation of differential equation. The two roots r_1 and r_2 are called indicial Roots.

So if r is known, then solving $A_1 = 0, A_2 = 0, A_3 = 0$ .... the unknown constant coefficient a_n's can be found out.

If $y_1(x)$ and $y_2(x)$ be two linearly independent solution of differential equation (2) Then the general solution of equation (2) is given by $y = Ay_1(x) + By_2(x)$. Where A and B are arbitrary constants.

Case - 1.

Roots not differing by an Integer : If $r_1 - r_2 \neq p$ is an integer. Hence the roots are distinct by putting $r = r_1$ and $r = r_2$ respectively, the complete solution is given by

$y = Ay_1(x) + By_2(x)$, with $y_1(x), y_2(x)$ obtained from equation $y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$ by putting $r = r_1$ and $r = r_2$.

Case - 2.

When the Roots are equal : i.e $r_1 - r_2 = 0$ or $r_1 = r_2 = m$

Then the two linearly independent solution are given by

$$y_1(x) = y(x)\big|_m \text{ and } y_2(x) = \left(\frac{\partial y}{\partial r}\right)_m$$

So $$y = A\{y(x)\}_m + B\left(\frac{\partial y}{\partial r}\right)_m$$

Case - 3.

When roots r_1 and r_2 are distinct and differ by integer :

If one of the coefficients becomes infinite for one of the roots then replace $a_0 = b_0(r - r_1)$.

$a_0 \neq 0$ one of the solutions is given by $y_1(x) = y(x)\big|_{r=r_1}$ and solution is given by $y_2(x) = \frac{\partial y}{\partial r}\Big|_{r=r_1}$

The solution is $y = A(y)_{r_1} + B\left(\frac{\partial y}{\partial r}\right)_{r_1}$

Example - 1 : Using Frobenius Method, solve in series the equation $4xy'' + 2y' + y = 0$.

Solution. The given equation is $4xy'' + 2y' + y = 0$

or $$y'' + \frac{2y'}{4x} + \frac{y}{4x} = 0 \text{ i.e } y'' + \frac{y'}{2x} + \frac{y}{4x} = 0$$ in standard from.

Here $x = 0$ is a regular singular point.

as $xp(x) = x\frac{1}{2x} = \frac{1}{2}$ and $x^2Q(x) = x^2\frac{1}{4x} = \frac{x}{4}$ are both analytic at $x = 0$. So we use frobenius method to obtain series solution about $x = 0$.

Let $\Rightarrow y = \sum_{r=0}^{\infty} ax^{k+r}$ .... (1)

$\therefore \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$ and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a(k+r)(k+r-1)x^{k+r-2}$ .... (2)

$\therefore$ Using equation (1), (2) and (3) in equation $4xy'' + 2y' + y = 0$ we get :

$$4x\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2} + 2\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} + \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow 4\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-1} + 2\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} + \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r(k+r)\{4(k+r-1)+2\}x^{k+r-1} + \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum a_r(k+r)\{4(k+r-1)+2\}x \ldots + \sum_{r=0} a_r \ldots$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r(k+r)(4k+4r-2)x^{k+r-1} + \sum_{r=0}^{\infty} a_r x^{k+r} = 0 \qquad \ldots\ldots\ldots (4)$$

Now equating to zero, the coefficient x^{k-1} (lowest power of x) (in the 1st summation only), we get :

Putting $r = 0$, $\boxed{a_0 k(4k-2) = 0}$ which is Indicial equation.

$\Rightarrow \boxed{k = 0}$ or $\boxed{4k - 2 = 0}$ as $a_0 \neq 0$

i.e $\boxed{k = 0}$ or $4k = 2$

$$\Rightarrow \boxed{k = \frac{2}{4} = \frac{1}{2}}$$

Equating to zero the coefficient of next lowest power of x i.e x^k(by putting $r = 1$ in 1st summation and putting $r = 0$ in 2n summation), we get :

$$a_1(k+1)(4k+4-2) + a_0 = 0$$

$$\Rightarrow a_1(k+1)(4k+2) + a_0 = 0$$

$$\Rightarrow a_1 = -\frac{1}{(k+1)(4k+2)} a_0 \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

Now equating to zero the coefficient of x^{k+r}, we get

$$a_{r+1}(k+r+1)(4k+4r+4-2) + a_r = 0$$

$$\Rightarrow a_{r+1}(k+r+1)(4k+4r+2) = -a_r$$

$$\Rightarrow a_{r+1} = \frac{-a_r}{(k+r+1)(4k+4r+2)} \qquad \ldots\ldots\ldots\ldots (6) \text{ which is the Recurrence Relation}$$

$\therefore$ For $r = 0$, $a_1 = \dfrac{-a_0}{(k+1)(4k+2)}$, For $r = 1$, $a_2 = \dfrac{-a_1}{(k+2)(4k+6)}$

$$\Rightarrow a_2 = \frac{a_0}{(k+1)(4k+2)(k+2)(4k+6)}$$

$$\Rightarrow a_2 = \frac{a_0}{(k+1)(k+2)(4k+2)(4k+6)}$$

For $r = 2$, $a_3 = \dfrac{-a_2}{(k+3)(4k+10)} = \dfrac{-a_0}{(k+1)(k+2)(k+3)(4k+2)(4k+6)(4k+10)}$

For $k = 0$, $\quad a_1 = \dfrac{-a_0}{2}, a_2 = \dfrac{a_0}{(2)(2)(6)} = \dfrac{a_0}{24}$

$$a_3 = \frac{-a_0}{(2)(3)(2)(6)(10)} = \frac{-a_0}{720} \text{ and so on.}$$

For $k=\frac{1}{2}$, $\quad a_1=\frac{-a_0}{\left(\frac{3}{2}\right)(4)}=-\frac{a_0}{6}$

$$a_2=\frac{a_0}{\frac{3}{2}\times\frac{5}{2}\times4\times8}=\frac{a_0}{120}$$

$$a_3=\frac{-a_0}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot4\times8\times12}=\frac{-a_0}{420\times12} \text{ and so on.}$$

Hence for $k=0,\ y_1=\sum_{r=0}^{\infty}a_rx^r$.

i.e. $y_1=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots$

$$=a_0-\frac{a_0}{2}x+\frac{a_0}{24}x^2-\frac{a_0}{720}x^3+\ldots\ldots$$

$$=a_0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\ldots\ldots\right)$$

For $k=\frac{1}{2}$, the second solution is given by

$$y_2=\sum_{r=0}^{\infty}a_rx^{\frac{1}{2}+r}=a_0x^{\frac{1}{2}}+a_1x^{\frac{3}{2}}+a_2x^{\frac{5}{2}}+a_3x^{\frac{7}{2}}+\ldots\ldots$$

i.e $\frac{1}{2}=a_0x^{\frac{1}{2}}-\frac{a_0x^{\frac{3}{2}}}{6}+\frac{a_0}{120}x^{\frac{5}{2}}-\frac{a_0}{5040}x^{\frac{7}{2}}+\ldots\ldots$

i.e $y_2=a_0x^{\frac{1}{2}}\left(1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5090}+\ldots\ldots\right)$

The general solution is given by $y=Ay_1+By_2$

i.e. $y=Aa_0\left(1-\frac{1}{2}x+\frac{1}{24}x-\frac{1}{720}x^3+\ldots\ldots\right)+Ba_0x^{\frac{1}{2}}\left(1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5040}+\ldots\ldots\right)$

$\Rightarrow y=A_1\left[1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\ldots\ldots\right]+A_2x^{\frac{1}{2}}\left[1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\ldots\ldots\right]$

when $A_1=Aa_0$, $A_2=Ba_0$.

Example - 1 :

Solve the differential equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ by Frobenius Method. (n is a Fraction).

Solution. The given differential equation is $x^2y'' + xy' + (x^2 - n^2)y = 0$ .... (1)

When put in standard form, we get $y'' + \frac{xy'}{x^2} + \left(\frac{x^2 - n^2}{x^2}\right)y = 0$

Or $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\frac{x^2 - n^2}{x^2}\right)y = 0$

Hence $P(x) = \frac{1}{x}; Q(x) = \frac{x^2 - n^2}{x^2}$. So $x = 0$ is a regular singular point as $xP(x) = 1 \neq \infty$, $x^2Q(x) = x^2 - n^2 \neq \infty$, when $x = 0$

Hence series solution of the given equation is possible using Frobenius Method about $x = 0$.

Let $$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \text{............ (2)}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} \quad \text{............ (3)}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} \quad \text{............ (4)}$$

Using these values in equation (1) we get :

$$x^2\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} + x\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} + (x^2 - n^2)\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r-2} - \sum_{r=0}^{\infty} n^2 a_r x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r x^{k+r}\left[(k+r)(k+r-1) + (k+r) - n^2\right] + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r x^{k+r}\left[(k+r)\{k+r-1+1\} - n^2\right] + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r x^{k+r}\left\{(k+r)^2 - n^2\right\} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0 \quad \text{............ (5)}$$

Equating coefficient of x^k to zero we get : (putting $r = 0$)

$a_0(k^2 - n^2) = 0$ which is Indicial equation

$\Rightarrow k^2 - n^2 = 0$ as $a_0 \neq 0 \Rightarrow \boxed{k = \pm n}$

Equating to zero the coefficient of x^{k+1} we get

$$a_1\left[(k+1)^2-n^2\right]=0 \Rightarrow \boxed{a_1=0} \text{ as } (k+1)^2-n^2 \neq 0$$

Now Equating to zero the coefficient of x^{k+r+2}, we get

$$a_{r+2}\{(k+r+2)^2-n^2\}+a_r=0$$

$$\Rightarrow a_{r+2}=\frac{-a_r}{(k+r+2+n)(k+r+2-n)} \qquad \text{.......... (6)}$$

which is the recurrence Relation.

For $r=0$, $a_2=\dfrac{-a_0}{(k+2+n)(k+2-n)}$. For $r=1$, $a_3=\dfrac{-a_1}{(k+3+n)(k+3-n)}=0$

$\therefore a_1=a_3=a_5=\text{........}=0$ For $r=2$, $a_4=\dfrac{-a_2}{(k+4+n)(k+4-n)}$

Case - 1 : When k = n

Then $a_2=\dfrac{-a_0}{2(2+2n)}=\dfrac{-a_0}{2(2n+2)}$

$$a_4=\frac{-a_2}{(2n+4)(4)}=\frac{a_0}{2.4(2n+2)(2n+4)} \text{ and so on.}$$

Then solution is $y_1(x)=\sum_{r=0}^{\infty}a_r x^{n+r}=a_0x^n+a_1x^{n+1}+a_2x^{n+2}+a_3x^{n+3}+a_4x^{n+4}+\text{..........}$

i.e $y_1(x)=a_0x^n+0-\dfrac{a_0}{2(2n+2)}x^{n+2}+0+\dfrac{a_0}{2\cdot4(2n+2)(2n+4)}x^{n+4}+\text{..........}$

i.e $y_1(x)=a_0x^n\left[1-\dfrac{x^2}{2(2n+2)}+\dfrac{x^4}{2\cdot4(2n+2)(2n+4)}-\dfrac{x^6}{2\cdot4\cdot6(2n+2)(2n+4)(2n+6)}+\text{...}\right]$ .... (7)

Case - 2 : For k = –n

The solution is

$$y_2(x)=a_0x^{-n}\left[1-\frac{x^2}{2(2-2n)}+\frac{x^4}{2\cdot4(2-2n)(4-2n)}-\frac{x^6}{2\cdot4\cdot6(2-2n)(4-2n)(6-2n)}+\text{...}\right] \qquad \text{.......... (8)}$$

So the complete solution is given by $y = Ay_1 + By_2$.

Note : The given differential Equation is known as Bessel's differential equation of order n.

It's particular solution are called Bessels Functions or order n. In particular if $a_0=\dfrac{1}{2^n\Gamma(n+1)}$

nple 1 : Solve $x\frac{d^2y}{dx^2}-y=0$ in power series using Frobenius Method.

Solution. The equation is $x\frac{d^2y}{dx^2}-y=0 \Rightarrow \frac{d^2y}{dx^2}-\frac{y}{x}=0$

$x=0$ is regular singular point as $x\,P(x)\neq\infty$, $x^2Q(x)\neq\infty$ at $x=0$

Let $$y=\sum_{r=0}^{\infty}a_r x^{k+r} \qquad \text{.............. (1)}$$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1} \qquad \text{.............. (2)}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2} \qquad \text{.............. (3)}$$

So using ther values in equation $x\frac{d^2y}{dx^2}-y=0$ we get

$$x\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}-\sum_{r=0}^{\infty}a_r x^{k+r}=0$$

$$\Rightarrow\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-1}-\sum_{r=0}^{\infty}a_r x^{k+r}=0 \qquad \text{.............. (4)}$$

Equating to zero the coefficient of x^{k-1}, we get (by putting $r=0$ in 1st summation)

$$a_0(k)(k-1)=0\Rightarrow \boxed{k=0} \text{ and } k-1=0\Rightarrow \boxed{k=1},\ a_0\neq 0$$

Equating to zero the coefficient of x^{k+r} we get

$$a_{r+1}(k+r+1)(k+r+1-1)-a_r=0$$

$$\Rightarrow a_{r+1}(k+r+1)(k+r)=a_r \Rightarrow a_{r+1}=\frac{a_r}{(k+r)(k+r+1)} \qquad \text{.............. (5)}$$

Now putting $r=0$, $a_1=\frac{a_0}{k(k+1)}$, For $r=1$, $a_2=\frac{a_1}{(k+1)(k+2)}=\frac{a_0}{k(k+1)^2(k+2)}$

For $r=2$, $a_3=\frac{a_2}{(k+2)(k+3)}=\frac{a_0}{k(k+1)^2(k+2)^2(k+3)}$ and so on.

$\therefore$ Equating to zero, the coefficient of x^{k-1}, we get :

$$a_0(k)^2 = 0 \Rightarrow k^2 = 0 \Rightarrow \boxed{k = 0,0} \text{ as } a_0 \neq 0.$$

Equating to zero, the coefficient of x^{k+r}, we get :

$$a_{r+1}(k+r+1)^2 + a_r(k+r+2) = 0$$

$$\Rightarrow \boxed{a_{r+1} = \frac{-(k+r+2)}{(k+r+1)^2}a_r} \quad \text{......... (6)}$$

Putting $r = 0$, $a_1 = \dfrac{-(k+2)}{(k+1)^2}a_0$

For $r = 1$, $a_2 = \dfrac{-(k+3)}{(k+2)^2}a_1 = \dfrac{(k+3)}{(k+2)^2}\dfrac{(k+2)}{(k+1)^2}a_0$

$$\Rightarrow a_2 = \frac{(k+3)}{(k+1)^2(k+2)}a_0$$

For $r = 2$, $a_3 = \dfrac{-(k+4)}{(k+3)^2}a_2 = \dfrac{-(k+4)}{(k+3)^2}\dfrac{(k+3)}{(k+1)^2(k+2)}$

$$\Rightarrow a_3 = \frac{-(k+4)}{(k+1)^2(k+2)(k+3)} \text{ and so on}$$

14 / 32

So the solution is $y = \sum_{r=0}^{\infty} a_r x^{k+r} = a_0x^k + a_1x^{k+1} + a_2x^{k+2} + a_3x^{k+3} + a_4x^{k+4} + \text{.........}$

i.e $y = a_0x^k - \dfrac{(k+2)}{(k+1)^2}a_0x^{k+1} + \dfrac{(k+3)}{(k+1)^2(k+2)}a_0x^{k+2} - \dfrac{(k+4)a_0x^{k+3}}{(k+1)^2(k+2)(k+3)} + \text{.......}$

$$y = a_0x^k\left[1 - \frac{(k+2)}{(k+1)^2}x + \frac{(k+3)}{(k+1)^2(k+2)}x^2 - \frac{(k+4)x^3}{(k+1)^2(k+2)(k+3)} + \text{.........}\right] \quad \text{.....(7)}$$

The first solution is obtained by putting $k = 0$ in the above equation

i.e $y_1(x) = a_0\left[1 - \dfrac{2}{1^2}x + \dfrac{3}{1^2\cdot 2}x^2 - \dfrac{4}{1^2\cdot 2\cdot 3}x^3 + \text{.......}\right] = a_0\left[1 - 2x + \dfrac{3}{2!}x^2 - \dfrac{4}{3!}x^3 + \text{.........}\right]$ .... (8)

To obtain the second independent solution

We differentiate equation (7) partially with respect to k.

i.e $\dfrac{\partial y}{\partial k} = a_0x^k \ln x\left[1 - \dfrac{(k+2)}{(k+1)^2}x + \dfrac{(k+3)}{(k+1)^2(k+2)}x^2 - \dfrac{(k+4)x^3}{(k+1)^2(k+2)(k+3)} + \text{.......}\right]$

$+ a_0x^k\left[\dfrac{2(k+2)}{(k+1)^3}x - \dfrac{1}{(k+1)^2}x + \dfrac{1}{(k+1)^2(k+2)}x^2 - \dfrac{(k+3)}{(k+1)^2(k+2)^2}x^2 - \dfrac{2(k+3)}{(k+1)^3(k+2)}x^2\right.$

$$-\frac{1}{(k+1)^2(k+2)(k+3)}x^3+\frac{(k+4)}{(k+1)^2(k+2)^2(k+3)}x^3+\frac{(k+4)}{(k+1)^2(k+2)(k+3}$$

$$+\frac{2(k+4)}{(k+1)^3(k+2)(k+3)}x^3+.........\Bigg]$$

Then the second solution is given by

$$y_2=\left(\frac{\partial y}{dk}\right)_{k=0}=a_0\ln x\left[1-2x+\frac{3x^2}{2!}-\frac{4x^3}{3!}+.........\right]$$

$$+a_0\left[4x-x+\frac{x^2}{1^2\cdot 2}-\frac{3x^2}{1^2\cdot 2^2}-\frac{2\cdot 3}{1^3\cdot 2}x^2-\frac{x^3}{1^2\cdot 2\cdot 3}+\frac{4x^3}{1^2\cdot 2^2\cdot 3}+\frac{4x^3}{1^2\cdot 2\cdot 3^2}+\frac{2\cdot 4x^3}{1^3\cdot 2\cdot 3}\right.$$

$$\Rightarrow y_2(x)=y_1\ln x+a_0\left(3x-\frac{13}{4}x^2+\frac{31}{18}x^3-.........\right)$$

The general solutionis given by

$$y=A\left(1-2x+\frac{3x^2}{2!}-\frac{4x^3}{3!}+.....\right)+B\left[y_1\ln x+a_0\left(3x-\frac{13}{4}x^2+\frac{31}{18}x^3.........\right)\right]$$

When Roots are distinct, but differ by an Integer

In this case if roots differ by integer, then two linear independent solutions are ob putting the two values of the roots in y(x) as in 1st case. But if some of the coefficient of y infinite when $k = k_1$ then a_0 is replaced by $C_0(k-k_1)$ i.e., $a_0 = C_0(k-k_1)$ to produce we coefficients. The solution is given by $y_1(x)=[y(x)]_{k=k_0}$ and second linearly independen is given by $y_2(x)=\left(\frac{\partial y}{\partial k}\right)_{k=k_0}$ which always contains a logarithmic term. So complete

$$y=a_1(y)_{k_1}+a_2\left(\frac{\partial y}{\partial k}\right)_{k_1}.$$

Example 1 : Solve $x\frac{d^2y}{dx^2}-y=0$ in power series using Frobenius Method

Example 1 : Solve $x\dfrac{d^2y}{dx^2} - y = 0$ in power series using Frobenius Method.

Solution. The equation is $x\dfrac{d^2y}{dx^2} - y = 0 \Rightarrow \dfrac{d^2y}{dx^2} - \dfrac{y}{x} = 0$

$x = 0$ is regular singular point as $xP(x) \neq \infty$, $x^2Q(x) \neq \infty$ at $x = 0$

Let $$y = \sum_{r=0}^{\infty} a_r x^{k+r} \qquad \text{............. (1)}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} \qquad \text{............. (2)}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} \qquad \text{............. (3)}$$

So using ther values in equation $x\dfrac{d^2y}{dx^2} - y = 0$ we get

$$x\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} - \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-1} - \sum_{r=0}^{\infty} a_r x^{k+r} = 0 \qquad \text{............ (4)}$$

Equating to zero the coefficient of x^{k-1}, we get (by putting $r = 0$ in 1st summation)

$$a_0 (k)(k-1) = 0 \Rightarrow \boxed{k = 0} \text{ and } k - 1 = 0 \Rightarrow \boxed{k = 1}, a_0 \neq 0$$

Equating to zero the coefficient of x^{k+r} we get

$$a_{r+1}(k+r+1)(k+r+1-1) - a_r = 0$$

$$\Rightarrow a_{r+1}(k+r+1)(k+r) = a_r \Rightarrow a_{r+1} = \frac{a_r}{(k+r)(k+r+1)} \qquad \text{............... (5)}$$

Now putting $r = 0$, $a_1 = \dfrac{a_0}{k(k+1)}$. For $r = 1$, $a_2 = \dfrac{a_1}{(k+1)(k+2)} = \dfrac{a_0}{k(k+1)^2(k+2)}$

For $r = 2$, $a_3 = \dfrac{a_2}{(k+2)(k+3)} = \dfrac{a_0}{k(k+1)^2(k+2)^2(k+3)}$ and so on.

So $y=\sum_{r=0}^{\infty} a_r x^{k+r} = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + a_3 x^{k+3} + a_4 x^{k+4} + \ldots\ldots$

i.e $y = a_0 x^k + \frac{a_0}{k(k+1)} x^{k+1} + \frac{a_0}{k(k+1)^2(k+2)} x^{k+2} + \frac{a_0}{k(k+1)^2(k+2)^2(k+3)} x^{k+3} + \ldots$

or $y = a_0 x^k \left[1 + \frac{x}{k(k+1)} + \frac{x^2}{k(k+1)^2(k+2)} + \frac{x^3}{k(k+1)^2(k+2)^2(k+3)} + \ldots\ldots\right]$ ....(6)

For $k=1$, equation (6) gives 1st solution

$$y_1(x) = a_0 x\left[1 + \frac{x}{1\cdot 2} + \frac{x^2}{1\cdot 3\cdot 2^2} + \frac{x^3}{1\cdot 4\cdot 2^2\cdot 3^2} + \ldots\ldots\right] \qquad \ldots\ldots (7)$$

Putting $k=0$, in equation (6), the coefficient of x^2, x^4, x^6 .... become infinite. Hence we put $a_0 = c_0 k$ in equation (6), then

$$y = c_0 x^k\left[k + \frac{x}{k+1} + \frac{x^2}{(k+1)^2(k+2)} + \frac{x^3}{(k+1)^2(k+2)^2(k+3)} + \ldots\ldots\right]$$

$$\therefore \frac{\partial y}{\partial k} = c_0 x^k \ln x\left[k + \frac{x}{k+1} + \frac{x^2}{(k+1)^2(k+2)} - \frac{x^3}{(k+1)^2(k+2)^2(k+3)} + \ldots\ldots\right]$$

$$+ c_0 x^k\left[1 - \frac{x}{(k+1)^2} - \frac{x^2}{(k+1)^2(k+2)^2} - \frac{2x^2}{(k+2)(k+1)^3} - \frac{x^3}{(k+1)^2(k+2)^2(k+3)^2}\right.$$

$$\left. - \frac{2x^3}{(k+1)^3(k+2)^2(k+3)} - \frac{2x^3}{(k+2)^2(k+2)^3(k+3)} + \ldots\ldots\right]$$

Then we put $k=0$ in above equation, we get

$$y_2 = \left(\frac{\partial y}{\partial k}\right)_{k=0} = c_0 \ln x\left[\frac{x}{1} + \frac{x^2}{1^2\cdot 2} + \frac{x^3}{1^2\cdot 2^2\cdot 3} + \ldots\ldots\right]$$

$$+ c_0\left[1 - \frac{x}{1^2} - \frac{x^2}{1^2\cdot 2^2} - \frac{2x^2}{2\cdot 1^3} - \frac{x^3}{1^2\cdot 2^2\cdot 3^2} - \frac{2x^3}{1^3\cdot 2^2\cdot 3} - \frac{2x^3}{1^2\cdot 2^3\cdot 3} + \ldots\ldots\right]$$

Another case : Roots differ by Integer such that one or more coefficients are indeterminate for one of the roots of indicial equation (For $k = k_0$). Thus root produces the complete solution as contains two arbitrary constants.

LEGENDRE'S DIFFERENTIAL EQUATION

The differential equation of the form $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$ .... (1)

is known as Legendre's equation. This equation can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0$$

Clearly x = 0 is an ordinary point of differential equation (1).

The series solution in descending powers of x has physical Importance So Let the solution be

of the form $$y=\sum_{r=0}^{\infty}a_r x^{k-r} \qquad \text{.................. (2)}$$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(k-r)x^{k-r-1} \qquad \text{.................. (3)}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2} \qquad \text{.................. (4)}$$

Using these values in equation (1) we get :

$$(1-x^2)\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2}-2x\sum_{r=0}^{\infty}a_r(k-r)x^{k-r-1}+n(n+1)\sum_{r=0}^{\infty}a_r x^{k-r}=0$$

i.e $$\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2}-\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r}$$

$$-2\sum_{r=0}^{\infty}a_r(k-r)x^{k-r}+n(n+1)\sum_{r=0}^{\infty}a_r x^{k-r}=0$$

i.e $$\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2}+\sum_{r=0}^{\infty}\left[n(n+1)-2(k-r)-(k-r)(k-r-1)\right]a_r x^{k-r}=0$$

or $$\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2}+\sum_{r=0}^{\infty}\left[n(n+1)-(k-r)\{2+k-r-1\}\right]a_r x^{k-r}=0$$

or $$\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2}+\sum_{r=0}^{\infty}\left[n(n+1)-(k-r)(k-r+1)\right]a_r x^{k-r}=0 \qquad \text{............ (5)}$$

Equating the coefficient of x^k to zero by putting r = 0 in the 2nd summation we get

$$a_0[n(n+1)-k(k+1)]=0$$

$\Rightarrow n(n+1)-k(k+1)=0$ as $a_0 \neq 0$ (as it is the 1st term)

i.e $n^2+n-k^2-k=0 \quad \Rightarrow (n^2-k^2)+(n-k)=0 \Rightarrow (n-k)(n+k)+(n+k)=0$

i.e $(n-k)(n+k+1)=0$ then $\boxed{k=n}$ or $\boxed{k=-(n+1)}$

Now equating the coefficients of x^{k-1} to zero by putting r = 1 we get

$$[n(n+1) - k(k-1)]a_1 = 0 \quad \text{i.e } [n^2 + n - k + k]a_1 = 0$$

$$\text{i.e} \quad [(n^2 - k^2) + (n + k)]a_1 = 0 \quad \text{i.e } [(n-k)(n+k) + (n+k)]a_1 = 0$$

$$\text{i.e} \quad (n+k)(n-k+1)a_1 = 0 \quad \Rightarrow \boxed{a_1 = 0} \text{ as } (n+k)(n-k+1) \neq 0$$

Equating the coefficient of x^{k-r-2} to zero we get :

$$(k-r)(k-r-1)a_r + [n(n+1) - (k-r-2)(k-r-1)]\, a_{r+2} = 0$$

$$\text{i.e} \quad a_{r+2} = -\frac{(k-r)(k-r-1)a_r}{n(n+1)-(k-r-2)(k-r-1)} \quad \text{....... (6)}$$ But $a_1 = 0$, $a_3 = a_5 = a_7 = ... = 0$

Case 1 : When k = n, equation (6) becomes

$$a_{r+2} = \frac{-(n-r)(n-r-1)}{n(n+1)-(n-r-2)(n-r-1)}a_0 \quad \text{............... (7)}$$

For r = 0, we get

$$a_2 = \frac{-n(n-1)a_0}{n(n+1)-(n-2)(n-1)} = \frac{-n(n-1)a_0}{n^2+n-(n^2-n-2n+2)} = \frac{-n(n-1)a_0}{(4n-2)}$$

$$\text{or} \quad a_2 = \frac{-n(n-1)}{2(2n-1)}a_0$$

$$\text{For } r = 2,\ a_4 \frac{-(n-2)(n-3)}{n(n+1)-(n-4)(n-3)}a_2 = \frac{-(n-2)(n-3)a_2}{n^2+n-(n^2-3n-4n+12)}a_2$$

$$\text{i.e } a_4 = \frac{-(n-2)(n-3)}{8n-12}a_2 = \frac{-(n-2)(n-3)}{4(2n-3)}a_2$$

$$\text{i.e } a_4 = \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-3)}a_0 \quad \text{(after putting value of } a_2)$$

In the similar manner we get for r = 4

$$a_6 = \frac{-n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-1)(2n-3)(2n-5)} \text{ and so on.}$$

So using the values of various coefficients a_1, a_2, a_3 .... in the solution $y = \sum_{r=0}^{\infty} a_r x^{k-r}$ for k = n we get.

$$y = \sum_{r=0}^{\infty} a_r x^{n-r} = = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + a_4 x^{n-4} + a_5 x^{n-5} + a_6 x^{n-6} + \text{.............}$$

$$\text{or} \quad y = a_0 x^n + 0 - \frac{n(n-1)}{2(2n-1)} a_0 x^{n-2} + 0 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-3)} a_0 x^{n-4} + 0$$

$$- \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-1)(2n-3)(2n-5)} a_0 x^{n-6} + \ldots\ldots\ldots$$

$$\text{or} \quad y = a_0 \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} x^{n-4} \right.$$

$$\left. - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-1)(2n-3)(2n-5)} x^{n-6} + \ldots\ldots\ldots \right] \quad \ldots\ldots(8)$$

But if $a_0 = \dfrac{1\cdot 3\cdot 5 \ldots\ldots\ldots (2n-1)}{n!}$ then the solution represented by equation (8) is known as Legendre's polynomial or Legendre's Function of first kind and is represented by $P_n(x)$.

$$\text{Then } P_n(x) = \frac{1\cdot 3\cdot 5 \ldots\ldots\ldots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)} x^{n-4} + \ldots \right] \quad \ldots\ldots (9)$$

Case 2. When $k = -(n+1)$

So equation (6) gives $a_{r+2} = \dfrac{-(-n-1-r)(-n-1-r-1)}{n(n+1)-(-n-1-r-2)(-n-1-r-1)} a_r$

$$\text{i.e} \quad a_{r+2} = \frac{-(n-r-1)(-n-r-2)}{n(n+1)-(-n-r-3)(-n-r-2)} a_r$$

$$\text{or} \quad a_{r+2} = \frac{-(n+r+1)(n+r+2)}{n(n+1)-(n+r+2)(n+r+3)} a_r \quad \ldots\ldots\ldots (10)$$

For $r = 0$, equation (10) gives $a_2 = \dfrac{-(n+1)(n+2)}{n(n+1)-(n+2)(n+3)} a_0$

$$\text{i.e} \quad a_2 = \frac{-(n+1)(n+2)}{n^2+n-n^2-5n-6} a_0 = \frac{-(n+1)(n+2)}{-2(2n+3)} a_0$$

$$\text{i.e} \quad a_2 = \frac{(n+1)(n+2)}{2(2n+3)} a_0$$

For $r = 2$, equation (10) gives $a_4 = \dfrac{-(n+3)(n+4)}{n(n+1)-(n+4)(n+5)} a_2$

i.e $$a_4 = \frac{-(n+3)(n+4)}{n^2+n-n^2-9n-20}a_2 = \frac{-(n+3)(n+4)}{-4(2n+5)}a_2$$

$$= \frac{(n+3)(n+4)(n+1)(n+2)}{2(2n+3)(2n+5)4}a_0 \qquad \text{(Using value of } a_2)$$

i.e $$a_4 = \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4(2n+3)(2n+5)}a_0$$

Similarly $a_6 = \dfrac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{2\cdot 4\cdot 6(2n+3)(2n+5)(2n+7)}a_0$ and so on.

Hence for $k = -(n + 1)$ the solution is given by :

$y = \sum_{r=0}^{\infty} a_r x^{k-r} = \sum_{r=0}^{\infty} a_r x^{-(n+1)-r}$ becomes

$$y = a_0x^{-n-1} + a_1x^{-n-2} + a_2x^{-n-3} + a_3x^{-n-4} + a_4x^{-n-5} + a_5x^{-n-6} + a_6x^{-n-7} + \ldots\ldots$$

$$= a_0x^{-n-1} + 0 + \frac{(n+1)(n+2)}{2(2n+3)}a_0x^{-n-3} + 0 + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}a_0x^{-n-5} + \ldots\ldots$$

or $$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}x^{-n-5} + \ldots\ldots\right]$$

$$\{\because a_1 = a_3 = a_5 = a_7 = \ldots\ldots = 0\} \qquad \ldots\ldots (11)$$

However if $a_0 = \dfrac{n!}{1\cdot 3\cdot 5\ldots\ldots(2n+1)}$ then above solution is called Legendre polynomial or Legendre Function of second kind and is denoted by $Q_n(x)$.

$$\text{So } Q_n(n) = \frac{n!}{1\cdot 3\cdot 5\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n++4)}{2\cdot 4(2n+3)(2n+5)}x^{-n-5} + \ldots\right]$$

$$\ldots\ldots (12)$$

Here $P_n(x)$ and $Q_n(x)$ are two independent solution of Legendre's equation, the most general solution of Legendre's equation is given by

$$y = AP_n(x) + BQ_n(x)$$

where A and B are arbitrary constants.

Generating Function for $P_n(x)$:

... are arbitrary constants.

enerating Function for P (x) :

RECURRENCE FORMULAE FOR LEGENDRE'S POLYNOMIALS

Using the Generating function for Legendre's polynomials, we can obtain Recurrence formulae also known as Recursion Relations which play important roles in solving problems, derivation etc.

1. $nP_n(x) = (2n -)xP_{n-1}(x) - (n-1)\,P_{n-1}(x)$

Solution. Using the Generating function for $P_n(x)$ we have :

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Differentiating above equation with respect to t we get : ....(1)

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-\frac{1}{2}-1}(-2x+2t) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x)$$

Or $$\left(1-2xt+t^2\right)^{-\frac{3}{2}}(x-t) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x)$$

Multiplying both sides of equation (2) by $(1 - 2xt + t^2)$ we have ....(2)

$$\left(1-2xt+t^2\right)^{-\frac{3}{2}+}(x-t) = \sum_{n=0}^{\infty} nt^{n-1}\left(1-2xt+t^2\right)P_n(x)$$

Or $$(x-t)\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} nt^{n-1}P_n(x) - \sum_{n=0}^{\infty} 2x\,nt^nP_n(x) + \sum_{m=0}^{\infty} nt^{n+1}P_n(x)$$

Or $$(x-t)\sum_{n=0}^{\infty} t^nP_n(x) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x) - \sum_{n=0}^{\infty} 2x\,nt^nP_n(x) + \sum_{n=0}^{\infty} nt^{n+1}P_n(x)$$

Or $$\sum_{n=0}^{\infty} xt^nP_n(x) - \sum_{n=0}^{\infty} t^{n+1}P_n(x) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x) - 2\sum_{n=0}^{\infty} nxt^nP_n(x) + \sum_{n=0}^{\infty} nt^{n+1}P_n(x)$$

Now Equating the coefficient of t^{n-1} on both sides we get :

$$xP_{n-1}(x) - P_{n-2}(x) = nP_n(x) - 2(n-1)xP_{n-1}(x) + (n-2)P_{n-2}(x)$$

i.e $nP_n(x) = xP_{n-1}(x) + 2(n-1)xP_{n-1}(x) - P_{n-2}(x) - (n-2)P_{n-2}(x)$

Or $nP_n(x) = xP_{n-1}(x)[1 + 2n - 2] - P_{n-2}(x)\,[1 + n - 2]$

Or $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$ .... (4 [illegible] above equation so that we get

[illegible]

[illegible]

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

Solution From the Generating function

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \ldots(1)$$

Differentiating equation (1) with respect to t, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} nt^{n-1} P_n(x)$$

$$\text{Or} \quad (x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} nt^{n-1} P_n(x) \quad \ldots(2)$$

Also differentiating equation (1) with respect to x, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} t^n P'_n(x) \quad \text{where } P'_n(x) = \frac{dP_n(x)}{dx}$$

$$\text{Or} \quad t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} t^n P'_n(x) \quad \ldots(3)$$

Dividing equation (2) by equation (3) we get

$$\frac{(x-t)(1-2xt+t^2)^{-3/2}}{(t)(1-2xt+t^2)^{-3/2}} = \frac{\sum_{n=0}^{\infty} nt^{n-1} P_n(x)}{\sum_{n=0}^{\infty} t^n P'_n(x)}$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} t^n P'_n(x) = \sum_{n=0}^{\infty} nt^{n-1} t P_n(x)$$

$$\text{i.e} \quad \sum_{n=0}^{\infty} xt^n P'_n(x) - \sum_{n=0}^{\infty} t^{n+1} P'_n(x) = \sum_{n=0}^{\infty} nt^n P_n(x)$$

Equating coefficient of t^n on both sides we get

$$xP'_n(x) - P'_{n-1}(x) = nP_n(x)$$

From Rodrigue's Formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

For $n = 0$, $P_0(x) = \frac{1}{2^0 0!} = 1$

For $n = 1$, $P_1(x) = \frac{1}{2^1 1!}\frac{d}{dx}(x^2-1) = \frac{1}{2}2x = x$

For $n = 2$, $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}[2(x^2-1)(2x)]$

i.e. $P_2(x) = \frac{1}{2}[(x^2-1)\cdot 1 + x\cdot 2x]$

i.e. $P_2(x) = \frac{1}{2}[3x^2-1]$

Similarly $P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$ and so on.

i.e. in general $P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$

Here $N = \frac{n}{2}$ if n is even But $N = \frac{n-1}{2}$, if n is odd.

... a_0, a_1, a_2 and a_3 in equation (1) we get

$$f(x) = 4x^3 - 2x^2 - 3x + 8 = \frac{22}{3}P_0(x) - \frac{3}{5}P_1(x) - \frac{4}{3}P_2(x) + \frac{8}{5}P_3(x)$$

...mple 3 : Express x^4 in terms of Legendre's polynomials.

Solution. We have $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

i.e $8P_4(x) = 35x^4 - 30x^2 + 3$

$$8P_4(x) + 30x^2 - 3 = 35x^4 \text{ i.e } x^4 = \frac{8P_4(x)}{35} + \frac{30}{35}x^2 - \frac{3}{35}$$

Also $P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow 2P_2(x) = 3x^2 - 1 \Rightarrow 3x^2 = 1 + 2P_2(x)$

or $x^2 = \frac{1}{3} + \frac{2}{3}P_2(x)$

So $x^4 = \frac{8}{35}P_4(x) + \frac{30}{35}\left[\frac{1}{3} + \frac{2}{3}P_2(x)\right] - \frac{3}{35}$

$$= \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{10}{35} - \frac{3}{35}$$

or $x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{7}{35}P_0(x)$ as $P_0(x) = 1$.

...mple 4 : Express $5P_4(x) + 10P_3(x) + 2P_2(x) + P_1(x)$ in term of power of x.

Solution. We have $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_1(x) = x.$$

Then $5P_4(x) + 10P_3(x) + 2P_2(x) + P_1(x)$

$$= 5.\frac{1}{8}(35x^4 - 30x^2 + 3) + 10\frac{1}{2}(5x^3 - 3x) + 2\cdot\frac{1}{2}(3x^2 - 1) + x$$

$\Rightarrow P_n(-x) = (-1)^n P_n(x).$

RECURRENCE FORMULAE FOR LEGENDRE'S POLYNOMIALS

Using the Generating function for Legendre's polynomials, we can obtain Recurrence formu also known as Recursion Relations which play important roles in solving problems, derivation e

1. $nP_n(x) = (2n -)xP_{n-1}(x) - (n-1)\, P_{n-1}(x)$

Solution. Using the Generating function for $P_n(x)$ we have :

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x) \qquad \text{.......(}$$

Differentiating above equation with respect to t we get :

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-\frac{1}{2}-1}(-2x+2t) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x)$$

$$\text{Or} \quad \left(1-2xt+t^2\right)^{-\frac{3}{2}}(x-t) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x) \qquad \text{.......}$$

Multiplying both sides of equation (2) by $(1 - 2xt + t^2)$ we have

$$\left(1-2xt+t^2\right)^{-\frac{3}{2}+}(x-t) = \sum_{n=0}^{\infty} nt^{n-1}\left(1-2xt+t^2\right)P_n(x)$$

$$\text{Or} \quad (x-t)\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} nt^{n-1}P_n(x) - \sum_{n=0}^{\infty} 2x\,nt^nP_n(x) + \sum_{n=0}^{\infty} nt^{n+1}P_n(x)$$

$$\text{Or} \quad (x-t)\sum_{n=0}^{\infty} t^nP_n(x) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x) - \sum_{n=0}^{\infty} 2x\,nt^nP_n(x) + \sum_{n=0}^{\infty} nt^{n+1}P_n(x)$$

$$\text{Or} \quad \sum_{n=0}^{\infty} xt^nP_n(x) - \sum_{n=0}^{\infty} t^{n+1}P_n(x) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x) - 2\sum_{n=0}^{\infty} nxt^nP_n(x) + \sum_{n=0}^{\infty} nt^{n+1}P_n(x)$$

Now Equating the coefficient of t^{n-1} on both sides we get :

$xP_{n-1}(x) - P_{n-2}(x) = nP_n(x) - 2(n-1)xP_{n-1}(x) + (n-2)P_{n-2}(x)$

i.e $nP_n(x) = xP_{n-1}(x) + 2(n-1)xP_{n-1}(x) - P_{n-2}(x) - (n-2)P_{n-2}(x)$

Or $nP_n(x) = xP_{n-1}(x)[1 + 2n - 2] - P_{n-2}(x)\,[1 + n - 2]$ ....

Or $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$

$\ldots(x)$, and $(2n+1)P_n = P'_{n+1}(x) - P'_{n-1}(x)$

ORTHOGONAL PROPERTY OF LEGENDRE'S POLYNOMIAL

The Orthogonal property of Legendre's polynomial is given by

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad \text{for } m \neq n$$

$$= \frac{2}{2n+1} \qquad \text{for } m = n$$

But using kronecker delta system δ_{mn} the orthogonality property of Legendre's polynomial is represented as

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \left(\frac{2}{2n+1}\right)\delta_{mn} \text{ where } \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

Proof. Legendre's differential equation is given by

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Or, $(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0$ .... (1)

As $P_n(x)$ and $P_m(x)$ are solutions of Legendre's equation (1), we have

$$\left(1-x^2\right)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \qquad \text{............ (2)}$$

And $\left(1-x^2\right)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0$ .... (3)

Now multiplying equation (2) by $P_m(x)$ and equation (3) by $P_n(x)$ and then subtracting we get

$$\left(1-x^2\right)\left[P_m''(x)P_n(x) - P_n''(x)P_m(x)\right] - 2x\left[P_m'(x)P_n(x) - P_n'(x)P_m(x)\right] \qquad \text{........(4)}$$
$$+ [m(m+1) - n(n+1)]P_m(x)\,P_n(x) = 0$$

But $\frac{d}{dx}\left[\left(1-x^2\right)\left\{P_m'(x)P_n(x) - P_m(x)P_n'(x)\right\}\right]$

$$= \left(1-x^2\right)\left[P_m''(x)P_n(x) + P_m'(x)P_n'(x) - P_m'(x)P_n'(x) - P_m(x)P_n''(x)\right]$$
$$+ \left[P_m'(x)P_n(x) - P_n'(x)P_m(x)\right][-2x]$$

$$\Rightarrow \frac{1}{2t}\left[\ln(1+2t+t^2)-\ln(1-2t+t^2)\right]=\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}P_n^2(x)\,dx$$

$$\Rightarrow \frac{1}{2t}\left[\ln(1+t)^2-\ln(1-t)^2\right]=\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}P_n^2(x)\,dx$$

$$\Rightarrow \frac{2}{2t}\left[\ln(1+t)-\ln(1-t)\right]=\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}P_n^2(x)\,dx$$

$$\Rightarrow \frac{1}{t}\left[\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\ldots\right)-\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-\ldots\right)\right]=\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}P_n^2(x)\,dx$$

$$\Rightarrow \frac{1}{t}\left[2t+\frac{2t^3}{3}+\frac{2t^5}{5}+\ldots\right]=\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}P_n^2(x)\,dx$$

$$\Rightarrow 2\left[1+\frac{t^2}{3}+\frac{t^4}{5}+\ldots+\frac{t^{2n}}{2n+1}+\ldots\right]=\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}P_n^2(x)\,dx \qquad (7)$$

Comparing the coefficients of t^{2n} on both sides we get

$$\int_{-1}^{1}P_n^2(x)\,dx=\frac{2}{2n+1} \qquad (8)$$

So the Orgonality property of Legendre's polynomial is given by

$$\int_{-1}^{1}P_m(x)P_n(x)\,dx=0 \qquad \text{if} \quad m\neq n$$

$$=\frac{2}{2n+1} \qquad \text{if} \quad m=n$$

Example 1 : Prove that $P'_n(1)=n(n+1)/2$

Solution. From Legendre's differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

But $P_n(x)$ is solution of above e equation

Then $(1-x^2)P''_n(x)-2xP'_n(x)+n(n+1)P_n(x)=0$

Now putting $x=1$ above equation becomes $-2P'_n(1)+n(n+1)P_n(1)=0$

i.e., $2P'_n(1)=n(n+1)P_n(1)$

i.e., $P'_n(1)=\dfrac{n(n+1)}{2}$ as $P_n(1)=1$

ample 2 : Show that $\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$

Solution. We know that the recurrence relation

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

i.e., $P_{n+1}(x) = \frac{(2n+1)}{(n+1)}xP_n(x) - \frac{n}{n+1}P_{n-1}(x)$

i.e., $xP_n(x) = \frac{(n+1)}{(n+1)}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$

Now Integrating both sides of above equation by $P_{n-1}(x)$ and Integrating the result w.r.t.
espect to x between limits −1 and 1, we get :

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{(n+1)}{(2n+1)}\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)dx + \frac{n}{(2n+1)}\int_{-1}^{1}[P_{n-1}(x)]^2 dx \quad \ldots$$

But from orthogonal property of Legendre's polynomial we know

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0 \qquad \text{if } m \neq n$$

$$= \frac{2}{2n+1} \qquad \text{if } m = n$$

Hence $\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)dx = 0$ ....(

And $\int_{-1}^{1}[P_{n-1}(x)]^2 dx = \frac{2}{2(n-1)+1} = \frac{2}{2n-1}$ ....(

Using equation (2) and (3) in equation (1) we get :

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = 0 + \frac{n}{(2n+1)}\frac{2}{(2n-1)} = \frac{2n}{4n^2-1}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

HERMITE'S DIFFERENTIAL EQUATION

The differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

where n is constant is called Hermite's differential equation. .... (1)

Let the series solution of equation (1) be given by

$$y = \sum_{r=0}^{\infty} a_r x^{k+r}$$ .... (2)

$$\therefore \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$$ .... (3)

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r-1)(k+r) x^{k+r-2}$$ .... (4)

$\therefore$ Putting the value of y, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (1) we get :

$$\sum_{r=0}^{\infty} a_r (k+r-1)(k+r) x^{k+r-2} - 2x\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} + 2n\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

Or, $$\sum_{r=0}^{\infty} a_r (k+r-1)(k+r) x^{k+r-2} - 2\sum_{r=0}^{\infty} a_r (k+r) x^{k+r} + 2n\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

Or, $$\sum_{r=0}^{\infty} a_r (k+r-1)(k+r) x^{k+r-2} - 2\left[\sum_{r=0}^{\infty} a_r (k+r) x^{k+r} - \sum_{r=0}^{\infty} n a_r x^{k+r}\right] = 0$$

Or, $$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} - 2\sum_{r=0}^{\infty} a_r (k+r-n) x^{k+r} = 0$$ .... (5)

Equation (5) is an Identity. To obtain the Indicial equation we equate to zero the coefficient of lowest power of x, i.e., x^{k-2}(by putting r = 0 in 1st summation).

i.e., $a_0 k(k-1) = 0$ or $k(k-1) = 0$ as $a_0 \neq 0$

Then $\boxed{k=0}$ or $\boxed{k=1}$

Now let us equate to zero the coefficient of x^{k-1} in equation (5) then $a_1(k+1)k = 0$

So $a_1 = 0$ when k = 1 and a_1 may or may not be zero when k = 0.

Again equating the coefficient of x^{k+r-2} to zero, we get :

$$a_r(k+r)(k+r-1) - 2a_{r-2}(k+r-2-n) = 0$$

$$\Rightarrow a_r(k+r)(k+r-1) = 2a_{r-2}(k+r-2-n)$$

$$\Rightarrow a_r = \frac{2(k+r-2-n)}{(k+r)(k+r-1)} a_{r-2}$$

$$\Rightarrow a_{r+2} = \frac{2(k+r+2-2-n)}{(k+r+2)(k+r+2-1)} a_{r+2-2}$$

$$\Rightarrow \boxed{a_{r+2} = \frac{2(k+r)-2n}{(k+r+2)(k+r+1)} a_r}$$

Case 1 : When k = 0

From equation (6) we get $a_{r+2} = \frac{(2r-2n)}{(r+2)(r+1)} a_r$

$\therefore$ **Putting r = 0, 2, 4 .... we get :**

$$a_2 = -\frac{2n}{2\cdot 1} a_0 = \frac{-2n}{2!} a_0$$

$$a_4 = \frac{4-2n}{4\cdot 3} a_2 = \frac{(4-2n)}{4\cdot 3}\left(-\frac{2n}{2!} a_0\right)$$

$$= \frac{2(2-n)}{4\cdot 3}\left(-\frac{2n}{2!} a_0\right)$$

$$= \frac{2^2 n(n-2)}{4!} a_0 \text{ and so on.}$$

In general $a_{2m} = \frac{(-2)^m n(n-2)\,.........\,(n-2m+2)}{2m!} a_0$

Putting r = 1, 3, 5 .... in equation (7) we get :

$$a_3 = \frac{2-2n}{3\cdot 2} a_1 = \frac{-2(n-1)}{3!} a_1$$

$$a_5 = \frac{6-2n}{5\cdot 4} a_4 = \left(\frac{6-2n}{5\cdot 4}\right)\left\{\frac{-2(n-1)}{3}\right\} a_1$$

$$= \frac{-2(3-n)}{5\cdot 4}\,\frac{2(n-1)}{3} a_1 = \frac{(-2)^2 (n-1)(n-3)}{5!} a_1$$

and so on.

5 MARK

1. Show that $\int_{-1}^{1} P_n(x)\,dx = 0$ for $n \neq 0$

2. Show that $y = \frac{d^n}{dx^n}(x^2-1)^n$ satisfies Legendre's differential Equation.

3. Show that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ is a solution of Legendre's differential Equation.

4. Show that $xP'_n(x) = P'_{n+1}(x) - (n-1)P_n(x)$.

5. Prove that $\int_{-1}^{1} xP_n(x)P_{n+1}(x)\,dx = \frac{2(n+1)}{(2n+1)(2n+3)}$

6. Prove that $\int_{-1}^{1} (x^2-1)P_{n+1}P'_n\,dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$.

7. Using Rodrigue's formula prove that
$$\int_{-1}^{1} x^m P_n(x)\,dx = 0 \text{ for } m<n.$$

8. Prove that
$$\int_{-1}^{1} P_n(x)(1-2tx+t^2)^{-\frac{1}{2}}\,dt = \frac{2t^n}{2n+1}$$

9. Show that $P'_{n+1} + P'_n = P_0 + 3P_1 + 5P_2 + 7P_3 + \cdots (2n+1)P_n$.

10. Show that $\int_{-\infty}^{\infty} x^2 e^{-x^2}\{H_n(x)\}^2\,dx = \sqrt{\pi}\,2^n n!\left(n+\frac{1}{2}\right)$

11. Prove that $\int_{-\infty}^{\infty} e^{-x^2/2} H_{2n+1}(x)\,dx = 0$

12. Prove that $H_{2n}(0) = (-1)^n 2^{2n}\left(\frac{1}{2}\right)$

13. Prove that $H'_{2n+1}(0) = (-1)^n 2^{2n+1}\left(\frac{3}{2}\right)$

14. Convert the Hermite polynomial $2H_3 - 4H_2 + H_1 + H_0$ into ordinary polynomial

1 MARK

1. $y'' + (x^2+3)y' + (x^3+3x^2+2x)y = 0$

2. $xy'' + y' + xy = 0$

3. $x'y'' + 3x^3y^3 + 2y = 0$

4. $x(x-1)^3y'' + 2(x-1)^3 + 4y = 0$

5. $e^x y'' + 4y' - 2xy = 0$

6. $x^2y'' + xy' + (x^2-n^2)y = 0$

7. $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$

$y \qquad = 0$

3 MARK

(a) Prove that for all x

1. $x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)$

2. $x^4 = \frac{1}{5} P_0(x) + \frac{4}{7} P_2(x) + \frac{8}{35} P_4(x)$.

3. $x^5 = \frac{3}{7} P_1(x) + \frac{4}{9} P_3(x) + \frac{8}{63} P_5(x)$.

4. Using Generating function for Hermite polynomials, the value of $H_0(x)$, $H_2(x)$, $H_3(x)$, $H_4(x)$.

5. Express $4x^4 + 8x^3 + 4x^2 - x - 3$ in terms of Hermite polynomials.

2 MARK

1. $4xy'' + 2y' + y = 0$

2. $9x(1+x)y'' - 6y' + 2y = 0$

3. $2x^2(1+x^2)y'' - 3x(1+x^2)y' + 2y = 0$

4. $4x^2y'' - 8xy' + 5y = 0$

5. $x(x^2+2)\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6xy = 0$

6. $2x^2y' + xy' + (x^2-3)y = 0$

7. $(1-x^2)y'' + 2xy' + y = 0$

8. $x^2y'' - xy'' - xy' + y = 0$

9. $x^2y'' - x(1+x)y' + y = 0$.